ECCENTRIC CONNECTIVITY INDEX OF SOME SPECIAL MOLECULAR GRAPHS AND THEIR R-CORONA GRAPHS

Yun Gao¹ --- Li Liang² --- Wei Gao³

¹Department of Editorial, Yunnan Normal University, China
²,³School of Information Science and Technology, Yunnan Normal University, China

ABSTRACT
In this paper, we determine the eccentric connectivity index and augmented eccentric connectivity index of fan graph, wheel graph, gear fan graph, gear wheel graph and their r-corona graphs.

Keywords: Chemical graph theory, Organic molecules, Eccentric connectivity index, Augmented eccentric connectivity index, Fan graph, Wheel graph, Gear fan graph, Gear wheel graph, r-corona graph.

1. INTRODUCTION
Wiener index, edge Wiener index, Hyper-wiener index and eccentric connectivity index are introduced to reflect certain structural features of organic molecules. Several papers contributed to determine the Wiener index or Hyper-wiener index of special graphs (See Yan, et al. [1], Gao and Shi [2] and Yan, et al. [3] for more detail). Let $P_n$ and $C_n$ be path and cycle with $n$ vertices. The graph $F_n = \{v\} \cup P_n$ is called a fan graph and the graph $W_n = \{v\} \cup C_n$ is called a wheel graph. Graph $I_r(G)$ is called $r$-crown graph of $G$ which splicing $r$ hang edges for every vertex in $G$. By adding one vertex in every two adjacent vertices of the fan path $P_n$ of fan graph $F_n$, the resulting graph is a subdivision graph called gear fan graph, denote as $\tilde{F}_n$. By adding one vertex in every two adjacent vertices of the wheel cycle $C_n$ of wheel graph $W_n$, The resulting graph is a subdivision graph, called gear wheel graph, denoted as $\tilde{W}_n$.

The graphs considered in this paper are simple and connected. The eccentricity $ec(u)$ of vertex $u \in V(G)$ is the maximum distance between $u$ and any other vertex in $G$. Then the eccentric connectivity index (ECI) of $G$ is defined as
\[ \xi^c(G) = \sum_{v \in V(G)} ec(v) \deg(v). \]

Ranjini and Lokesha [4] determined the eccentric connectivity index of the subdivision graph of the complete graphs, tadpole graphs and the wheel graphs. Morgan, et al. [5] obtained an exact lower bound on \( \xi^c(G) \) in terms of order, and showed that this bound is sharp. An asymptotically sharp upper bound was also derived. In addition, for trees of given order, when the diameter was also prescribed, precise upper and lower bounds are provided. Hua and Das [6] studied the relationship between the eccentric connectivity index and Zagreb indices. De [7] presented the explicit generalized expressions for the eccentric connectivity index and polynomial of the thorn graphs, and then considered some particular cases. Eskender and Vumar [8] calculated the eccentric connectivity index and eccentric distance sum of generalized hierarchical product of graphs. Moreover, they presented the exact formulae for the eccentric connectivity index of \( F \)-sum graphs in terms of some invariants of the factors. Iiic and Gutman [9] proved that the broom has maximum \( \xi^c(G) \) among trees with a fixed maximum vertex degree, and characterized such trees with minimum \( \xi^c(G) \). Iranmanesh and Hafezieh [10] presented the eccentric connectivity index of some graph families. Dankelmann, et al. [11] proved the upper bound for eccentric connectivity index and constructed graphs which asymptotically attain the bound. Morgan, et al. [12] showed that a known tight lower bound on the eccentric connectivity index for a tree, in terms of order and diameter, was also valid for a general graph, of given order and diameter. Rao and Lakshmi [13] obtained explicit formulas for eccentric connectivity index of phenylenic nanotubes.

The augmented eccentric connectivity index (AECI) \( \xi^A(G) \) of a molecular graph \( G \) is defined as

\[ \xi^A(G) = \sum_{v \in V(G)} \frac{M(v)}{ec(v)}, \]

where \( M(v) \) denotes the product of degrees of all neighbors of vertex \( v \).

determined the exact formulas for the augmented eccentric connectivity index of some special nanotube and nanotorus. Ediz [19] obtained an exact formula for the augmented eccentric connectivity index of an infinite class of nanostar dendrimers.

In this paper, we present the eccentric connectivity index of \( I_r(F_n) \), \( I_r(W_n) \), \( I_r(\tilde{F}_n) \) and \( I_r(\tilde{W}_n) \). Also, the augmented eccentric connectivity index of \( I_r(F_n) \), \( I_r(W_n) \), \( I_r(\tilde{F}_n) \) and \( I_r(\tilde{W}_n) \) are derived.

2. ECCENTRIC CONNECTIVITY INDEX

**Theorem 1.** \( \xi^e(I_r(F_n)) = r(7n + 5) + (11n - 6) \).

**Proof.** Let \( P_n = v_1v_2 \ldots v_n \) and the \( r \) hanging vertices of \( v_i \) be \( v_i^1, v_i^2, \ldots, v_i^r \) \((1 \leq i \leq n)\). Let \( v \) be a vertex in \( F_n \) beside \( P_n \), and the \( r \) hanging vertices of \( v \) be \( v^1, v^2, \ldots, v^r \). By the definition of eccentric connectivity index, we have

\[
\xi^e(I_r(F_n)) = ec(v) \deg(v) + \sum_{i=1}^{n} ec(v_i) \deg(v_i) + \sum_{i=1}^{r} ec(v^j) \deg(v^j) + \sum_{i=1}^{n} \sum_{j=1}^{r} ec(v^j_i) \deg(v^j_i)
\]

\[
= 2(r + n) + (3nr + (9n - 6)) + 3r + 4nr
\]

\[
= r(7n + 5) + (11n - 6). \Box
\]

**Corollary 1.** \( \xi^e(F_n) = 7n - 4 \).

**Theorem 2.** \( \xi^e(I_r(W_n)) = r(7n + 5) + 11n \).

**Proof.** Let \( C_n = v_1v_2 \ldots v_{n-1} \) and \( v_i^1, v_i^2, \ldots, v_i^r \) be the \( r \) hanging vertices of \( v_i \) \((1 \leq i \leq n)\). Let \( v \) be a vertex in \( W_n \) beside \( C_n \), and \( v^1, v^2, \ldots, v^r \) be the \( r \) hanging vertices of \( v \). By the definition of eccentric connectivity index, we have

\[
\xi^e(I_r(W_n)) = ec(v) \deg(v) + \sum_{i=1}^{n} ec(v_i) \deg(v_i) + \sum_{i=1}^{r} ec(v^j) \deg(v^j) + \sum_{i=1}^{n} \sum_{j=1}^{r} ec(v^j_i) \deg(v^j_i)
\]

\[
= 2(r + n) + (3nr + 9n) + 3r + 4nr
\]

\[
= r(7n + 5) + 11n.
\]
Corollary 2. \( \xi^c(W_n) = 7n \).

Theorem 3. \( \xi^c(I_r(\tilde{F}_n)) = r(20n - 4) + (25n - 18) \).

Proof. Let \( P_n = v_1v_2 \ldots v_n \) and \( v_{i+1} \) be the adding vertex between \( v_i \) and \( v_{i+1} \). Let \( v_i^1, v_i^2, \ldots, v_i^r \) be the \( r \) hanging vertices of \( v_i \) (\( 1 \leq i \leq n \)). Let \( v_{i+1}^1, v_{i+1}^2, \ldots, v_{i+1}^r \) be the \( r \) hanging vertices of \( v_{i+1} \) (\( 1 \leq i \leq n-1 \)). Let \( v \) be a vertex in \( F_n \) beside \( P_n \), and the \( r \) hanging vertices of \( v \) be \( v^1, v^2, \ldots, v^r \). By virtue of the definition of eccentric connectivity index, we get

\[
\xi^c(I_r(\tilde{F}_n)) = ec(v) \deg(v) + \sum_{i=1}^{n} ec(v_i) \deg(v_i) + \sum_{i=1}^{r} ec(v_i') \deg(v_i') + \sum_{i=1}^{n} \sum_{j=1}^{r} ec(v_i') \deg(v_i')
\]

\[
+ \sum_{i=1}^{n-1} ec(v_{i+1}) \deg(v_{i+1}) + \sum_{i=1}^{n-1} \sum_{j=1}^{r} ec(v_{i+1}) \deg(v_{i+1})
\]

\[
= 3(r + n) + (4nr + (12n - 8)) + 4r + 5nr + 5(n - 1)(2 + r) + 6r(n - 1)
\]

\[
= r(20n - 4) + (25n - 18) .
\]

Corollary 3. \( \xi^c(\tilde{F}_n) = 19n - 14 \).

Theorem 4. \( \xi^c(I_r(\tilde{W}_n)) = r(20n + 7) + 25n \).

Proof. Let \( C_n = v_1v_2 \ldots v_n \) and \( v \) be a vertex in \( W_n \) beside \( C_n \). \( v_{i+1} \) be the adding vertex between \( v_i \) and \( v_{i+1} \). Let \( v^1, v^2, \ldots, v^r \) be the \( r \) hanging vertices of \( v \) and \( v_i^1, v_i^2, \ldots, v_i^r \) be the \( r \) hanging vertices of \( v_i \) (\( 1 \leq i \leq n \)). Let \( v_{n+1} = v_{i+1} \) and \( v_{i+1}^1, v_{i+1}^2, \ldots, v_{i+1}^r \) be the \( r \) hanging vertices of \( v_{i+1} \) (\( 1 \leq i \leq n \)). In view of the definition of eccentric connectivity index, we deduce

\[
\xi^c(I_r(\tilde{W}_n)) = ec(v) \deg(v) + \sum_{i=1}^{n} ec(v_i) \deg(v_i) + \sum_{i=1}^{r} ec(v_i') \deg(v_i') + \sum_{i=1}^{n} \sum_{j=1}^{r} ec(v_i') \deg(v_i') +
\]

\[
\sum_{i=1}^{n} ec(v_{i+1}) \deg(v_{i+1}) + \sum_{i=1}^{n} \sum_{j=1}^{r} ec(v_{i+1}) \deg(v_{i+1})
\]

\[
= 3(r + n) + (4nr + 12n) + 4r + 5nr + 5n(2 + r) + 6nr
\]
\[ r(20n + 7) + 25n. \]

**Corollary 4.** \( \xi^c(\tilde{W}_n) = 19n. \)

### 3. AUGMENTED ECCENTRIC CONNECTIVITY INDEX

**Theorem 5.** 
\[ \xi^A(I_r(F_n)) = \frac{r^2(3n + 4) + r(35n - 8) + (4n^2 + 42n - 44)}{12}. \]

**Proof.** By the definition of augmented eccentric connectivity index, we have

\[ \xi^A(I_r(F_n)) = \frac{M(v) + \sum_{i=1}^{n} M(v_i) + \sum_{i=1}^{r} M(v_i') + \sum_{i=1}^{n} \sum_{j=1}^{r} M(v_i'j)}{ec(v)} = \frac{r(n+1) + (3n-2) + r(4n-2) + (n^2 + 6n - 8) + r(n+r) + n(3r+r^2) - 2r}{2} = \frac{r^2(3n+4) + r(35n-8) + (4n^2 + 42n-44)}{12}. \]

**Corollary 5.** 
\[ \xi^A(F_n) = \frac{n^2 + 12n - 12}{2}. \]

**Theorem 6.** 
\[ \xi^A(I_r(W_n)) = \frac{r^2(3n+4) + r(35n+6) + (4n^2 + 42n)}{12}. \]

**Proof.** By the definition of augmented eccentric connectivity index, we have

\[ \xi^A(I_r(W_n)) = \frac{M(v) + \sum_{i=1}^{n} M(v_i) + \sum_{i=1}^{r} M(v_i') + \sum_{i=1}^{n} \sum_{j=1}^{r} M(v_i'j)}{ec(v)} = \frac{r(n+1) + 3n + 4nr + (n^2 + 6n) + r(n+r) + n(3r+r^2)}{2} = \frac{r^2(3n+4) + r(35n+6) + (4n^2 + 42n)}{12}. \]

**Corollary 6.** 
\[ \xi^A(W_n) = \frac{n^2 + 12n}{2}. \]

**Theorem 7.** 
\[ \xi^A(I_r(\tilde{F}_n)) = \frac{r^2(22n+5) + r(187n-90) + (15n^2 + 192n - 196)}{60}. \]

**Proof.** By virtue of the definition of augmented eccentric connectivity index, we get

\[ \xi^A(I_r(\tilde{F}_n)) = \frac{M(v) + \sum_{i=1}^{n} M(v_i) + \sum_{i=1}^{r} M(v_i') + \sum_{i=1}^{n} \sum_{j=1}^{r} M(v_i'j) + \sum_{i=1}^{n-1} \sum_{j=1}^{r} M(v_i'j)}{ec(v)} = \frac{r^2(22n+5) + r(187n-90) + (15n^2 + 192n - 196)}{60}. \]
\[
\begin{align*}
&= \frac{r(n+1) + (3n-2)}{3} + \frac{r(4n-2) + (n^2 + 4n - 4)}{4} + \frac{r(n+r)}{4} + \frac{r^2 n + r(3n-2)}{5} + \\
&\quad \frac{r(3n-3) + (6n-8)}{5} + \frac{r(n-1)(2+r)}{6} \\
&= \frac{r^2 (22n+5) + r(187n-90) + (15n^2 + 192n - 196)}{60}.
\end{align*}
\]

**Corollary 7.** \( \xi^A(\bar{F}_n) = \frac{n^2 + 13n - 13}{3}. \)

**Theorem 8.** \( \xi^A(I_r(\tilde{W}_n)) = \frac{r^2 (22n+15) + r(187n+20) + (15n^2 + 192n)}{60}. \)

**Proof.** In view of the definition of augmented eccentric connectivity index, we deduce

\[
\begin{align*}
\xi^A(I_r(\tilde{W}_n)) &= \frac{M(v)}{ec(v)} + \sum_{i=1}^{n} \frac{M(v_i)}{ec(v_i)} + \sum_{i=1}^{r} \frac{M(v_i)}{ec(v_i)} + \sum_{i=1}^{n} \sum_{j=1}^{r} \frac{M(v_{i,j})}{ec(v_{i,j})} + \sum_{i=1}^{n} \frac{M(v_{i,r+1})}{ec(v_{i,r+1})} + \\
&\quad \sum_{i=1}^{n} \sum_{j=1}^{r} \frac{M(v_{i,j})}{ec(v_{i,j})} \\
&= \frac{r(n+1) + 3n + 4nr + (n^2 + 4n)}{3} + \frac{r(n+r)}{4} + \frac{n(3r + r^2)}{5} + \frac{3nr + 6n}{5} + \frac{rn(2+r)}{6} \\
&= \frac{r^2 (22n+15) + r(187n+20) + (15n^2 + 192n)}{60}.
\end{align*}
\]

**Corollary 8.** \( \xi^A(\tilde{W}_n) = \frac{n^2 + 13n}{3}. \)

### 4. ACKNOWLEDGEMENTS

First, we thank the reviewers for their constructive comments in improving the quality of this paper. This work was supported in part by the National Natural Science Foundation of China (61262071), and the Key Science and Technology Research Project of Education Ministry (210210). We also would like to thank the anonymous referees for providing us with constructive comments and suggestions.

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