ON HOMOGENEOUS CUBIC EQUATION WITH FOURUNKNOWN

\[ x^3 + y^3 = 21zw^2 \]

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ABSTRACT

The homogeneous cubic equation with four unknowns represented by the Diophantine equation \[ x^3 + y^3 = 21zw^2 \] is analyzed for its patterns of non–zero distinct integer solutions. A few interesting properties between the solutions and special numbers, namely, Polygonal number, Pyramidal number, Centered polygonal number, Stella octangular number and Octahedral number are presented.

Keywords: Homogeneous cubic, Cubic equation with four unknowns, Integral solutions, Cubic Diophantine equation, Third degree equation, Special numbers.

Notations Used

- Polygonal number of rank \( n \) with size \( m \).
  \[ t_{m,n} = n[1 + \frac{(n-1)(m-2)}{2}] \]
- Pyramidal number of rank \( n \) with size \( m \).
  \[ p_n^m = \frac{1}{6}[n(n+1)][(m-2)n + (5 - m)] \]
- Centered polygonal number of rank \( n \) with size \( m \).
  \[ ct_{m,n} = \frac{mn(n+1)+2}{2} \]
- Stella octangular number of rank \( n \)
  \[ SO_n = n(2n^2 - 1) \]
- Octahedral number of rank \( n \)
\[ \text{OH}_n = \frac{1}{3} n (2n^2 + 1) \]

**Contribution/ Originality**

This study contributes in the existing literature different approaches of determining non-zero distinct integer solutions to the homogeneous equation of degree three with 4 unknowns given by \( x^3 + y^3 = 21zw^2 \)

1. **INTRODUCTION**

Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity can be seen as in \([1-3]\). The Diophantine equations offer an unlimited field for research due to their variety. The problem of finding all integer solutions of a Diophantine equation with three or more variables and degree at least three, in general presents a good deal of difficulties. Cubic equation in three variables falls into the theory but is still an important topic of current research \([4-6]\). Equations with more than three variables and degree at least three are known very little.

In particular, one may refer \([7-18]\) for cubic equations with four unknowns. This research concerns with yet another interesting equation \( x^3 + y^3 = 21zw^2 \) representing the homogeneous cubic equation with four unknowns for determining its infinitely many non-zero integer points. Also a few interesting properties are presented.

2. **SOME INTERESTING PATTERNS**

The homogeneous cubic Diophantine equation with four unknowns to be solved is given by \( x^3 + y^3 = 21zw^2 \) \hspace{1cm} (1)

which is written as \( (x + y)(x^2 - xy + y^2) = 21zw^2 \) \hspace{1cm} (2)

Suppose \( z = x + y \) \hspace{1cm} (3)

Substitute (3) into (2), it reduces to the quadratic equation \( (x^2 - xy + y^2) = 21w^2 \) \hspace{1cm} (4)

Let \( x = u + v, y = u - v \) \hspace{1cm} (5)

where \( u \) and \( v \) are non-zero distinct arbitrary integers.

Substituting (5) in (4), it gives \( u^2 + 3v^2 = 21w^2 \) \hspace{1cm} (6)

Equation (6) is solved through different approaches and the different patterns of solutions of (1) obtained are presented below.
2.1 Pattern-1

Assume \( w = a^2 + 3b^2 = (a + i\sqrt{3}b)(a - i\sqrt{3}b) \) \hspace{1cm} (7)

Write 21 as \( 21 = (3 + i2\sqrt{3})(3 - i2\sqrt{3}) \) \hspace{1cm} (8)

Using (7) and (8) in (6), it is written as

\[
(u + i\sqrt{3}v)(u - i\sqrt{3}v) = (3 + i2\sqrt{3})(3 - i2\sqrt{3})(a + i\sqrt{3}b)^2(a - i\sqrt{3}b)^2
\]  

Suppose that following system of equations are derived from (9)

\[
(u + i\sqrt{3}v) = (3 + i2\sqrt{3})(a + i\sqrt{3}b)^2
\]

\[
(u - i\sqrt{3}v) = (3 - i2\sqrt{3})(a - i\sqrt{3}b)^2
\]

Equating the real and imaginary parts in either of the above two equations, we get

\[
u = 3a^2 - 9b^2 - 12ab
\]

\[
v = 2a^2 - 6b^2 + 6ab
\]

Hence, in view of (9) and (5), we have

\[
\begin{align*}
x &= x(a,b) = 5a^2 - 15b^2 - 6ab \\
y &= y(a,b) = a^2 - 3b^2 - 18ab \\
z &= z(a,b) = 6a^2 - 18b^2 - 24ab
\end{align*}
\]  

Thus (7) and (10) represent non-zero distinct integer solutions for (1).

Properties of pattern-1: It is easy to infer following properties from (10)

- \( x(n,2n^2 - 1) - 5y(n,2n^2 - 1) = 6y(n,2n^2 - 1) - z(n,2n^2 - 1) = 84n(2n^2 - 1) = 84SO_n \)
- \( z(n,n(n + 1)) - x(n,n(n + 1)) - w(n,n(n + 1)) - 9p_n^5 = 6t_{4,n} \)
- \( x(n,2n^2 + 1) - 5y(n,2n^2 + 1) = 84(n,2n^2 + 1) = 252OH_n \)
- \( x(n,n) - z(n,n) + w(n,n) - 8t_{4,n} = 16n^2 \), a perfect square
- \( x(n,n) - z(n,n) + w(n,n) = 24n^2 \), a nasty number.

Notes of pattern-1:
Instead of (8), write 21 as \[ 21 = \frac{(3 + i5\sqrt{3}) - (3 - i5\sqrt{3})}{4} \]

Following the procedure presented in pattern-1, the corresponding integer solutions of (1) are

\[
\begin{align*}
\begin{cases}
x = x(a, b) = 4a^2 - 12b^2 - 12ab \\
y = y(a, b) = -a^2 + 3b^2 - 18ab \\
z = z(a, b) = 3a^2 - 9b^2 - 30ab \\
w = w(a, b) = a^2 + 3b^2
\end{cases}
\]

2.2 Pattern-2

Equation (6) can be written as \( u^2 + 3v^2 = 21w^2 \) \(*1\) \quad (11)

Write 1 as \( 1 = \frac{(1 + i\sqrt{3})(1 - i\sqrt{3})}{4} \) \quad (12)

Using (7), (8) and (12) in (11), it is written as

\[
(u + i\sqrt{3}v)(u - i\sqrt{3}v) = (3 + i2\sqrt{3})(3 - i2\sqrt{3})(a + i\sqrt{3}b)^2(a - i\sqrt{3}b)^2 \left(\frac{1 + i\sqrt{3}}{2}\right) \left(\frac{1 - i\sqrt{3}}{2}\right)
\]

Consider \( (u + i\sqrt{3}v) = (3 + i2\sqrt{3})(a + i\sqrt{3}b)^2 \left(\frac{1 + i\sqrt{3}}{2}\right) \)

Equating real and imaginary parts, we have

\[
\begin{align*}
u &= \frac{1}{2} \left[ -3a^2 + 9b^2 - 30ab \right] \\
v &= \frac{1}{2} \left[ 5a^2 - 15b^2 - 6ab \right]
\end{align*}
\]

Substituting the above value of \( u \) and \( v \) in (3) and (5), we obtain

\[
\begin{cases}
x = x(a, b) = a^2 - 3b^2 - 18ab \\
y = y(a, b) = -4a^2 + 12b^2 - 12ab \\
z = z(a, b) = -3a^2 + 9b^2 - 30ab
\end{cases}
\]

Thus (7) and (12a) represent non-zero distinct integer solutions for (1).

Properties of pattern-2: It is easy to infer following properties from (12a)

- \( x(n, -n) + w(n, -n) = 20n^2 = 20t_{4,n} \)
\[ 4x(n, n + 1) + y(n, n + 1) = -84n(n + 1) = 3x(n, n + 1) + z(n, n + 1) \]

\[ 3w(n, n) - z(n, n) = 36n^2 \text{ is a perfect square} \]

\[ 3\{x(n, 2n^2 - 1) + w(n, 2n^2 - 1) - 18SO_n\} = 6n^2, \text{ a nasty number} \]

\[ 21^2\{17x(\text{ } -n, n^2) + y(\text{ } -n, n^2) + z(\text{ } -n, n^2)\} = (42n)^3, \text{ a cubical integer.} \]

**2.3 Pattern-3**

Instead of (12), write 1 as \[ 1 = \frac{(1 + i4\sqrt{3})(1 - i4\sqrt{3})}{49} \]

Repeating the above process as in pattern-2, the non-zero distinct integral solutions of (1) are found to be

\[ x = x(a, b) = -a^2 + 3b^2 - 18ab \]
\[ y = y(a, b) = -5a^2 + 15b^2 - 6ab \]
\[ z = z(a, b) = -6a^2 + 18b^2 - 24ab \]
\[ w = w(a, b) = a^2 + 3b^2 \]

**Properties of pattern-3:** It is easy to infer following properties from above equations

\[ 5x(n^2, n + 1) - y(n^2, n + 1) = -84(n^2, n + 1) \equiv 0(\text{mod} 84) \]

\[ 3x(n, n) - y(n, n) + 165t_{4, n} = 81n^2, \text{ a perfect square} \]

\[ w(n, n) + x(n, n) + \text{ perfect square} = n^2 = t_{4, n} \]

\[ 6x(n, 2n^2 + 1) - z(n, 2n^2 + 1) = 252OH_n \]

**2.4 Pattern-4**

One may write (6) as \( u^2 - 9w^2 = 3(4w^2 - v^2) \)

Write (13) in the form of ratio as

\[ \frac{u + 3w}{3(2w - v)} = \frac{2w + v}{u - 3w} = \frac{a}{b}, b \neq 0 \]

Which is equivalent to the system of double equations
Applying the method of cross-multiplication, we have

\[ u = 9a^2 - 3b^2 + 12ab \]
\[ v = 6a^2 - 2b^2 - 6ab \]
\[ w = 3a^2 + b^2 \]

Hence, in view of (3) and (5), the corresponding values of x, y and z are given by

\[
\begin{align*}
\text{x} &= x(a, b) = 15a^2 - 5b^2 + 6ab \\
\text{y} &= y(a, b) = 3a^2 - b^2 + 18ab \\
\text{z} &= z(a, b) = 18a^2 - 6b^2 + 24ab
\end{align*}
\]

Thus (15) and (15a) represent non-zero distinct integer solutions for (1).

Properties of pattern-4: It is easy to infer following properties from (15a)

- \( 5y(n,2n^2 - 1) - x(n,2n^2 - 1) - 2SO_n = 82(n,2n^2 - 1) \equiv 0 \text{(mod } 82) \)
- \( x(n, n + 1) + y(n, n + 1) - z(n, n + 1) = 0 \)
- \( y(n, 19n^2 - 13) + w(n, 19n^2 - 13) - 108CP_{19,n} - t_{14,n} = 5n \equiv 0 \text{(mod } 5) \)
- Each of the following represents a perfect square
  - \( 6\{y(n,19n^2 - 13) + w(n,19n^2 - 13) - 108CP_{19,n}\} = 36n^2 \)
  - \( 21\{6y(n,n) - z(n,n)\} = (42n)^2 \)

2.5 Pattern-5

Equation (6) can be written as

\[ 3v^2 = 21w^2 - u^2 \]  \hfill (16)

Write \( v = 21a^2 - b^2 \)  \hfill (17)

Write \( s \) as

\[ s = \frac{(\sqrt{21} + 3)(\sqrt{21} - 3)}{4} \]  \hfill (17a)

Substituting (17) and (17a) in (16), we get
\[(\sqrt{21}w + u)(\sqrt{21}w - u) = (\sqrt{21}a + b)^2(\sqrt{21}a - b)^2\left(\frac{\sqrt{21} + 3}{2}\right)\left(\frac{\sqrt{21} - 3}{2}\right)\]

Consider

\[(\sqrt{21}w + u) = (\sqrt{21}a + b)^2\left(\frac{\sqrt{21} + 3}{2}\right)\]

Equating rational and irrational parts, we have

\[u = \frac{1}{2}[63a^2 + 3b^2 + 42ab]\]

and

\[w = \frac{1}{2}[21a^2 + b^2 + 6ab]\]  \quad (18)

Replacing \(a\) by \(2A\) and \(b\) by \(2B\) in (18), and using (3), (5), (17) and (17a), we have

\[x = x(A, B) = 210A^2 + 2B^2 + 24AB\]

\[y = y(A, B) = 42A^2 + 10B^2 + 84AB\]

\[z = z(A, B) = 252A^2 + 12B^2 + 168AB\]

\[w = w(A, B) = 42A^2 + 2B^2 + 12AB\]

**Properties of pattern-5:**

- \(6[y(n, n^2) - w(n, n^2)] - 5y(n, n^2) + x(n, n^2) = 96n^3 = 96CP_{6,n}\)
- \(z(n, n + l) - 6w(n, n + l) = 96n(n + l) = 6[y(n, n + l) - w(n, n + l)] - [5y(n, n + l) - x(n, n + l)]\)
- \(y(n, n) - w(n, n) = 80n^2 \equiv 0(\text{mod} 80)\)
- \(y(n, n) - w(n, n) + t_{4,A} = 8l_2^n\), a perfect square
- \(z(n, n) - 6w(n, n) = 96n^2 = \text{a Nasty number}\)

**3. CONCLUSION**

In this paper, we have illustrated different ways of obtaining non-zero distinct integer solutions to the homogeneous cubic equation with four unknowns given by \(x^3 + y^3 = 21zw^2\).

As the Diophantine equations are rich in variety, one may search for the integral solutions of other forms of cubic Diophantine equations along with their corresponding properties.

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