The main objective of this paper is to find the approximate solutions of the Black-Scholes (BS) model by two numerical techniques, namely, Du Fort-Frankel finite difference method (DF3DM), and Galerkin weighted residual method (GWRM) for both (call and put) type of European options. Since both DF3DM and GWRM are the most familiar numerical techniques for solving partial differential equations (PDE) of parabolic type, we estimate options prices by using these techniques. For this, we first convert the Black-Scholes model into a modified parabolic PDE, more specifically, in DF3DM, the first temporal vector is calculated by the Crank-Nicolson method using the initial boundary conditions and then the option price is evaluated. On the other hand, in GWRM, we use piecewise modified Legendre polynomials as the basis functions of GWRM which satisfy the homogeneous form of the boundary conditions. We may observe that the results obtained by the present methods converge fast to the exact solutions. In some cases, the present methods give more accurate results than the earlier results obtained by the adomian decomposition method [14]. Finally, all approximate solutions are shown by the graphical and tabular representations.

Contribution/Originality: The paper’s primary contribution is finding that the approximate results of Black-Scholes model by DF3DM, and GWRM with modified Legendre polynomials as basis functions.

1. INTRODUCTION

Options are treated as the most important part of the security markets from the beginning of the Chicago Board Options Exchange (CBOE) in 1973, which is the largest options market in the world [11]. During last decades, the valuation of options has become important problem for both financial and mathematical point of view. Details about options are available in Hull [1]; Privault [2]. There are many models for calculating the value of options but among all of those models, the Black-Scholes model is a suitable way to find the European options price.

calculated European call option values by using Black-Scholes formula. In Jódar, et al. [7] Black-Scholes model was solved by using Mellin transformation without numerical experiment. In Boyle [8] Monte Carlo method has been used for calculating the value of options with the accuracy and reliability. Very recently, Hok and Chan [9] develop a European option pricing method by using Fourier series with Legendre polynomials. In Carr and Madan [10] Peter Carr and Dilip B. Madan have shown the fast Fourier transform with the values of options.

Let us consider, \( F(S, t) \) = option price, \( S \) = stock price, \( K \) = strike price, \( \sigma \) = volatility, \( T \) = maturity time, \( r \) = risk-free interest rate and \( t \) = time in years. Then the famous Black-Scholes PDE is

\[
\frac{\partial F}{\partial t} + rS \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - rF = 0, \quad 0 < S < \infty, \quad 0 < t < T
\]  

(1)

1.1. Terminal and Boundary Conditions

1.1.1. Conditions for European Call Options

Terminal condition: \( F(S, t) = \text{max}(S - K, 0) \) at \( t = T \)

Boundary conditions: \( F(S, t) = 0 \), when \( S = 0 \) and \( F(S, t) = S - Ke^{-r(T-t)} \), when \( S \to \infty \)

1.1.2. Conditions European Put Options

Terminal condition: \( F(S, t) = \text{max}(K - S, 0) \) at \( t = T \)

Boundary conditions: \( F(S, t) = Ke^{-r(T-t)} \), when \( S = 0 \) and \( F(S, t) = 0 \), when \( S \to \infty \)

1.2. Black-Scholes Option Pricing Formula

The price of European call option, \( C = S\phi(d_+) - Ke^{-r(T-t)} \phi(d_-) \)  

(2)

The price of European put option, \( P = -S\phi(-d_+) + Ke^{-r(T-t)} \phi(-d_-) \)  

(3)

where,  

\[ d_+ = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}, \quad d_- = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \]

\[ \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy \]

1.3. Transformation of Black-Scholes PDE into heat Equation for Approximating Option Price

Substituting \( y = \ln(S/K), \tau = \frac{\sigma^2}{2}(T-t) \) and \( u(y, \tau) = \frac{1}{\kappa} e^{-\alpha y + \beta \tau} F(Ke^{y}, T - \frac{2\tau}{\alpha^2}) \) into equation \( 1 \), we get

\[
\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial y^2}; \quad \alpha \leq y \leq b; \quad 0 \leq \tau \leq T
\]  

(4)
where $S$ varies from $S_{\text{min}}$ to $S_{\text{max}}$ and $y$ varies from $\alpha \left( \ln \left( \frac{S_{\text{min}}}{K} \right) \right)$ to $b \left( \ln \left( \frac{S_{\text{max}}}{K} \right) \right)$.

Again substituting $y = a + (b - a)x$ into equation (4), we get

$$\frac{\partial u}{\partial t} = \frac{1}{(b-a)^2} \frac{\partial^2 u}{\partial x^2}; \quad 0 \leq x \leq 1; \quad 0 \leq t \leq \frac{2}{\sigma_i^2}$$  (5)

where, $u(x, \tau) = \frac{1}{\kappa} e^{-[\alpha(x + (b-a)x) + \beta \tau]} \Phi \left( K \sqrt{\alpha(x + (b-a)x)}, T - \frac{2x}{\sigma_i^2} \right)$  (6)

$$\theta = \frac{2x}{\sigma_i^2}, \quad \alpha = \frac{1}{4} (\theta - 1), \quad \text{and} \quad \beta = \frac{1}{4} (\theta + 1)^2$$

1.4. Transformation of Terminal and Boundary Conditions

1.4.1. Transformed Conditions for European Call Options

Initial condition: $u(x, 0) = e^{-\alpha x} \max \left( e^{(a+b)x} - 1, 0 \right)$ when $0 < x < 1$

Boundary condition: $u(0, \tau) = 0$ and $u(1, \tau) = e^{(1-a)b-\beta \tau} - e^{-ab+a^2 \tau}$

1.4.2. Transformed Conditions for European Put Options

Initial condition: $u(x, 0) = e^{a+b-x} \max \left( 1 - e^{a+b-x}, 0 \right)$ when $0 < x < 1$

Boundary condition: $u(0, \tau) = e^{-\alpha x + a^2 \tau}$ and $u(1, \tau) = 0$

1.5. Formulation of Du Fort–Frankel Finite Difference Method (DF3DM)

The Du Fort–Frankel finite difference method [11] can be applied to solve different kinds of problems that often occur in finance. This method is a two-step method and require another method for calculating first temporal vector. In this formulation, $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$ are approximated by central differencing and $u_i^j$ is replaced by $\frac{u_i^{j+1} + u_i^{j-1}}{2}$.

Thus, discretizing (5) by DF3DM, we get

$$\frac{u_i^{j+1} - u_i^{j-1}}{2\Delta \tau} = \frac{1}{(b-a)^2} \frac{u_i^{j-1} - 2u_i^{j+1} + u_i^{j+1}}{(\Delta x)^2} + u_i^{j+1}$$

Or, $u_i^{j+1} = u_i^{j-1} + \frac{2\Delta \tau}{(\Delta x (b-a))^2} \left( u_i^{j-1} - u_i^{j+1} - u_i^{j-1} + u_i^{j+1} \right)$
or, \[ u^{i+1}_t = u^{i-1}_t + 2d(u^{i}_t - u^{i+1}_t - u^{i-1}_t + u^{i+1}_t), \]
where \( d = \frac{\Delta \tau}{(\Delta x(b-a))^2} \)

\[ \therefore u^{i+1} = \frac{2d}{1+2d} (u^{i-1}_t + u^{i+1}_t) + \frac{1-2d}{1+2d} u^{i}_t \]  

(7)

where, \( i = 0, 1, \ldots, n - 1 \) and \( j = 0, 1, \ldots, m - 1 \)

Equation (7) can be written in matrix form as follows

\[
\begin{bmatrix}
    u^{i+1}_1 \\
    u^{i+1}_2 \\
    \vdots \\
    u^{i+1}_{n-1} \\
    u^{i+1}_n
\end{bmatrix} =
\begin{bmatrix}
    0 & \frac{2d}{1+2d} & 0 & \cdots & 0 \\
    \frac{2d}{1+2d} & 0 & \frac{2d}{1+2d} & 0 & \vdots \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & \frac{2d}{1+2d} & \cdots & 0 & \frac{2d}{1+2d} \\
    0 & \cdots & 0 & \frac{2d}{1+2d} & 0
\end{bmatrix} +
\begin{bmatrix}
    u^{i}_1 \\
    u^{i}_2 \\
    \vdots \\
    u^{i}_{n-1} \\
    u^{i}_n
\end{bmatrix}
\]

1.6. Formulation of Galerkin Weighted Residual Method (GWRM)

The Galerkin weighted residual method [12] is a well-known numerical technique for solving parabolic type PDE. By this method, we can convert a continuous operator problem into a discrete problem.

Let us consider the following parabolic initial boundary value problem:

\[
\mu(x) \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left[ a(x) \frac{\partial u}{\partial x} \right] + \beta(x) u(x,t) = f(x,t) \]  

(8)

Domain: \( x_{\text{min}} \leq x \leq x_{\text{max}} \); \( t \geq t_0 \)

Boundary Conditions: \( u(x_{\text{min}}, t) = u_{x_{\text{min}}} (t), u(x_{\text{max}}, t) = u_{x_{\text{max}}} (t); t > t_0 \)

Initial Condition: \( u(x, t_0) = u_0(x); x_{\text{min}} < x < x_{\text{max}} \)
Suppose a trial solution of (8) is 

\[ \tilde{u}(x,t) = \sum_{j=1}^{n} a_j(t) \varphi_j(x) \]  

(8a)

where each of the functions \(\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)\) satisfy the corresponding homogeneous form of the boundary conditions. Now the residual function of (8) is

\[ R(x,t) = \mu(x) \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left[ \alpha(x) \frac{\partial u}{\partial x} \right] + \beta(x) \tilde{u}(x,t) - f(x,t) \]  

(8b)

So, \( R(x,t) = 0 \) if and only if \( \tilde{u}(x,t) \) coincides with \( u(x,t) \) for all \( x \) and \( t \) in the domain of (8). In this method, we ascertain the parameters \( a_1(t), a_2(t), \ldots, a_n(t) \) by solving \( n \) equations

\[ \int_{x_{\text{min}}}^{x_{\text{max}}} R(x,t) \varphi_i(x) \, dx = 0, \quad i = 1, 2, \ldots, n \]  

(8c)

These equations are known as Galerkin equations and the functions \( \varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x) \) are called basis functions. Applying (8a), (8b) into (8c) we get

\[ \sum_{j=1}^{n} \left[ \int_{x_{\text{min}}}^{x_{\text{max}}} \mu(x) \varphi_i(x) \varphi_j(x) \, dx \right] \frac{da_j}{dt} + \sum_{j=1}^{n} \left[ \int_{x_{\text{min}}}^{x_{\text{max}}} \alpha(x) \frac{d \varphi_i}{dx} \frac{d \varphi_j}{dx} \, dx \right] a_j(t) \]

\[ + \sum_{j=1}^{n} \left[ \int_{x_{\text{min}}}^{x_{\text{max}}} \beta(x) \varphi_i(x) \varphi_j(x) \, dx \right] a_j(t) \]

\[ = \int_{x_{\text{min}}}^{x_{\text{max}}} f(x,t) \varphi_i(x) \, dx + \left[ (\alpha(x) \frac{\partial \tilde{u}}{\partial x}) \varphi_i(x) \right]_{x=x_{\text{min}}}^{x=x_{\text{max}}} \]

which can be written as the following matrix form

\[ [C] \left\{ \frac{da(t)}{dt} \right\} + [K] \{ u(t) \} = [F(t)] \]

(9)

where, \( C_{ij} = \int_{x_{\text{min}}}^{x_{\text{max}}} \mu(x) \varphi_i(x) \varphi_j(x) \, dx \),

\( K_{ij} = \int_{x_{\text{min}}}^{x_{\text{max}}} \alpha(x) \frac{d \varphi_i}{dx} \frac{d \varphi_j}{dx} \, dx + \int_{x_{\text{min}}}^{x_{\text{max}}} \beta(x) \varphi_i(x) \varphi_j(x) \, dx \)

and \( F_i(t) = \int_{x_{\text{min}}}^{x_{\text{max}}} f(x,t) \varphi_i(x) \, dx + \left[ (\alpha(x) \frac{\partial \tilde{u}}{\partial x}) \varphi_i(x) \right]_{x=x_{\text{min}}}^{x=x_{\text{max}}} \)
In the above notation \( C_{ij} \) and \( K_{ij} \) are called the heat capacity matrix and stiffness matrix respectively, whereas \( F \) is known as the load matrix.

Now applying equation (5) and (8) into (1), we get

\[
\mu(x) = 1, \quad \alpha(x) = \frac{1}{(b-a)^2}, \quad \beta(x) = 0, \quad f(x, \tau) = 0, \quad x_{\min} = 0 \text{ and } x_{\max} = 1
\]

Thus equation (1) reduces to

\[
[C]\left( \frac{d\omega(\tau)}{d\tau} \right) + [K]\{\alpha(\tau)\} = 0
\]  

(10)

where, \( C_{ij} = \int_0^1 \varphi_i(x) \varphi_j(x) dx, \quad K_{ij} = \frac{1}{(b-a)^2} \int_0^1 \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} dx \)

1.7. Modified Legendre Polynomials

The Legendre polynomials [13] of order \( n \) is,

\[
P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [(1 - x^2)^n], \quad n \geq 1
\]  

(11)

We make minor changes to the Legendre polynomials in such a way that each of the new polynomials satisfy the analogous homogeneous form of the boundary conditions of equation (5). We call those polynomials as Modified Legendre polynomials, \( \tilde{P}_n(x) \) and are defined as follows

\[
\tilde{P}_n(x) = x[P_n(x) - 1], \quad n \geq 1
\]  

(12)

Some Modified Legendre polynomials are given below:

\[
\begin{align*}
\tilde{P}_1(x) &= x(x - 1), \quad \tilde{P}_2(x) = \frac{3}{2} x(x^2 - 1), \quad \tilde{P}_3(x) = \frac{1}{2} x(5x^3 - 3x - 2), \\
\tilde{P}_4(x) &= \frac{5}{8} x(7x^4 - 6x^2 - 1), \quad \tilde{P}_5(x) = \frac{1}{8} x(63x^5 - 70x^3 + 15x - 6)
\end{align*}
\]
From Figure 1, we observe that all Modified Legendre polynomials gives 0 at x=0 and 1.

1.8. Numerical Experiments and Result Discussions

In this section, we take two data set for valuing European call option and one data set for finding the value of European put option.

To calculate option values by using DF3DM, we first put \( n = m = 250 \) (we can also choose \( m \neq n \)) in equation (7), then using (6) we get approximate values (here we use Crank-Nicolson method to calculate first temporal values). Moreover, approximating option values by using GWRM, we put Modified Legendre polynomials of order 1, 2, 3, and 4 as basis function in equation (10).

**Data set 1**: For approximating call option value we chose the parameters as follows [6]

\[ K = 50, \quad T = 0.5, \quad r = 0.10, \quad \sigma = 0.4, \quad \text{and } t = 0. \]

Now substituting the parameters given in data set 1 into equation (2), we get

\[
C(S_0) = S_0 \Phi(d_+) - 47.561471 \Phi(d_-)
\]  
where,

\[
d_+ = \frac{\ln\left(\frac{S_0}{K}\right)+0.09}{0.282843} \quad \text{and} \quad d_- = \frac{\ln\left(\frac{S_0}{K}\right)+0.01}{0.282843}
\]
Data set 2: For approximating put option value we chose the parameters as follows [4]:

\[
K = 10, \quad T = 1/3, \quad r = 0.10, \quad \sigma = 0.45, \quad \text{and} \quad t = 0.
\]

Now substituting the parameters given in data set 2 into equation (3), we get

\[
P(S_0) = -S_0\phi(-d_+) + 9.672161\phi(-d_-)
\]

where

\[
d_+ = \frac{\ln\left(\frac{S_0}{10}\right) + 0.067083}{0.259808} \quad \text{and} \quad d_- = \frac{\ln\left(\frac{S_0}{10}\right) - 0.000417}{0.259808}
\]
Figures 2 (a) and 3. (a), shows that approximate option prices obtained by DF3DM, nearly equal to BS values for all initial stock prices and from Figures 2. (b) and 3. (b) we observe that when the initial stock price is greater than the strike price then the approximate values obtained by GWRM fluctuates slightly from the BS values.

Figure 4 shows that, relative errors of DF3DM are minimum than relative errors of GWRM almost all the time.

Data set 3: \(\{S_0 = 65, K = 40, r = 0.05, \sigma = 0.324366, t = 0, T = \frac{1}{4}, \frac{1}{6} \text{ and } \frac{1}{12}\}\) [14]

Now substituting the parameters given in data set 3 into equation (2), we get

\[ C(T) = 65 \times \Phi(d_+ - 40 \times e^{-0.05 \times T} \Phi(d_-) \] (15)

where,

\[ d_+ = \frac{0.48551+0.10261 \times T}{0.324366 \times \sqrt{T}} \quad \text{and} \quad d_- = \frac{0.48551-0.0025067 \times T}{0.324366 \times \sqrt{T}} \]

From Table 1, we see that, most of the cases our proposed (present) methods give better results than Reference values.

<table>
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<tr>
<th>T</th>
<th>BS value</th>
<th>Present DF3DM Value</th>
<th>Present DF3DM Relative error</th>
<th>Present GWRM Value</th>
<th>Present GWRM Relative error</th>
<th>Reference Value</th>
<th>Reference Relative error</th>
</tr>
</thead>
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<td>25.5009</td>
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<td>1.11 \times 10^{-04}</td>
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<tr>
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<td>25.3325</td>
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<td>2.88 \times 10^{-09}</td>
<td>25.1663</td>
<td>7.95 \times 10^{-07}</td>
</tr>
</tbody>
</table>

2. CONCLUSIONS

From the results of this paper we observe that the results obtained by DF3DM is better than the results obtained by GWRM comparing with the exact (BS) and reference values. Finally, we can conclude that the DF3DM may be a good alternative to solve BS model for European options. Furthermore, we can improve the
approximations of these methods by increasing the value of $m$ and $n$ in DF3DM and increasing the number of basis functions in GWRM.

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