PROPERTIES OF PERMUTATION GROUPS USING WREATH PRODUCT

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ABSTRACT

Classification of the p-subgroups of the finite group of order 12 was done using Cauchy’s Lagrange’s and Sylow’s Theorems up to Isomorphism subgroups and related to the Dihedral group of order 20 \((D_{20})\) in Chemical Bonding.

Keywords: Isomorphism, Dihedral groups, Chemical bonding, Wreath product, Permutation group, Primitive and transitive.

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1. INTRODUCTION

In Mathematics, the Wreath Product of Group Theory is a specialized product of two groups, based on a semi-direct product. Wreath product is an important tool in the classification of permutation groups and also provides a way of constructing interesting examples of groups. The Wreath product of two groups \(A\) and \(B\) is written as \(A \wr B\). This is the standard wreath product, for other definitions see \([Arbib \[1\]; Hunter \[2\]; Kosheler \[3\] and Nakajima \[4\]]\). The wreath product of \(A\) and \(B\) contains the direct product \(A \times B\) as a sub-semi-group. If \(A\) has an identity, then any ideal extension of \(A\) by \(B\) can be imbedded in \(A \wr B\) \([5]\). The question of when \(A \wr B\) inherits various properties of \(A\) and \(B\) has been investigated mainly for various types of simplicity. Some examples are as follows; If \(A\) and \(B\) are completely-simple semi-groups and \(A\) is left-simple, then \(A \wr B\) is Completely-Simple \([5]\). If \(A\) and \(B\) are semi-groups, with Completely-Simple Kernels, thus \(A \wr B\) has a Completely-Simple Kernel \([9]\).

The Wreath product and its generalizations play an important role in the algebraic theory of automata. For example, they can be used to prove the theorem on the decomposition of every finite Semi-group automation into a step wise combination of flip-flope and Simple group automata \([2]\).

2. MATERIALS AND METHODS

2.1 The main objective of this paper is to study under which conditions the wreath products of permutation groups are faithful, transitive and primitive and present an example to support the findings.

2.2 Theorem and Definition: Let \(A\) and \(B\) be two permutation groups on \(\Gamma\) and \(\Delta\) respectively. Let \(A^\Delta\) be the set of all maps of \(\Delta\) into the permutation group, defined by \(A^\Delta := \{f: \Delta \to A\} \) for all \(f_1, f_2 \in A^\Delta\) and let \(A^\Delta\) be defined \(f\) or \(\delta \in \Delta\) by \((f_1 f_2)\delta = f_1(\delta_1) f_2(\delta_2)\) with respect to multiplication, \(A^\Delta\) assume the structure of a group.

Proof:
\((i)\) \(A^\delta\) is non-empty and is closed with respect to multiplication. For if \(f_1, f_2 \in A^\delta\), then \(f_1(\delta) f_2(\delta) \in A\). Since \(f_1(\delta) f_2(\delta) \in A\), this implies that \((f_1 f_2) \delta \in A\) and so \((f_1, f_2) \in A^\Delta\).

\((ii)\) Multiplication in \(A\) is associative. So also is the multiplication in \(A^\Delta\).

\((iii)\) The map \(\{e: \Delta \to A\}\) given by \(e(\delta) = 1\) for all \(\delta \in \Delta\) and \(1 \in A\) is the identity element in \(A^\Delta\).

\((iv)\) For every element \(f \in A^\Delta\) defined by all \(\delta \in \Delta\) by \(f^{-1}(\delta) = f(\delta)^{-1}\), thus \(A^\Delta\) is a group under multiplication and we call it \(G\).

**Lemma 2.3:** Suppose that \(B\) acts on \(G\) as follows 
\[ f^b(\delta) = f(\delta^{-1}b) \] for all \(\delta \in \Delta\), \(beB\), then \(B\) acts on \(G\) as a group.

**Proof:** Let \(f, f_1, f_2 \in G\) and \(b, b_1, b_2 \in B\), then \((f^b_1 b_2)(\delta) = (f^b_1 \delta) b_2^{-1}\) is non-empty. Since \((f(\delta^{-1}b_1)) f_2(\delta) = f(\delta^{-1}b_2)\delta = f(\delta^{-1})(\delta^{-1}) b_2^{-1}\) implies that \((f_1 f_2) \delta = (f_1 \delta f_2 \delta) f_2^{-1}\) is closed. This implies that \((f_1 f_2) \delta = \delta = f_2^{-1}(\delta) f_1(\delta)\) is associative. So also is the multiplication in \(A^\Delta\).

**Theorem 2.4:** Let \(B\) act on \(G\) as a group. Thus the set of ordered pairs \((f, b)\) with \(f \in G\) and \(b \in B\) is the group \(W\). Thus this implies that \((f^b)\) exists. And thus \(B\) acts on \(G\) as a group if the set of all ordered pairs \((f, b)\) if we define
\[ (f, b)(e, 1) = (f^b, 1) = (f, b) \] for all \(f, b \in G\), \(e \in B\).

**Definition 2.8:** The stabiliser of any point \((\alpha, \delta)\) under the action of \(W\) on \(\Delta\) defined \(\delta^B := \{\delta : b \in B\}\). For example, the orbit of 1 under the subgroup \(\{1, 2, 3\}\) of \(S_3\), is \(\{1, 2, 3\}\).

**Definition 2.8:** The stabiliser of any point \((\alpha, \delta)\) under the action of \(W\) on \(\Delta\) defined \(\delta^B := \{\delta : b \in B\}\).

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Definition 2.9: For any two points \((a_1, \delta_1)\) and \((a_2, \delta_2)\) in \(\Gamma \times \Delta\), then \(W\) will be transitive on \(\Gamma \times \Delta\) if and only if \(f b e W, f \in G, b \in B\) such that \((a_1, \delta_1)f b = (a_2, f(\delta_1), \delta, b) = (a_2, \delta_2)\). Thus such \(f, b\) exist if \(A\) and \(B\) are transitive on \(\Gamma\) and \(\Delta\) respectively which is a necessary condition for \(W\) to be transitive on \(\Gamma \times \Delta\).

Definition 2.10: \(W\) is faithful on \(\Omega\) if and only if the identity element of \(W\) is its only element that fixes every point of \(\Omega\) or \(W\) is faithful on \(\Omega\) if \(A\) and \(B\) are faithful on \(\Omega\) respectively.

Definition 2.11: \(W\) is primitive on \(\Omega\), if and only if given any \((\delta, b)\) in \(\Gamma \times \Delta\), the stabiliser of \((\delta, b)\) is a maximal subgroup of \(W\). That is, if \(W_{(\alpha, \delta)} \neq W\) and \(W_{(\alpha, \delta)} \Delta U_{(\alpha, \delta)} \Delta W\) implies either \(W_{(\alpha, \delta)} = U_{(\alpha, \delta)}\) or \(U_{(\alpha, \delta)} = W\).

Definition 2.12: Let \(G\) be a group acting on \(\Omega\), a block for \(G\) is a subset \(|\Omega| = 1\) but that for all \(g \in G, \Omega^0 \cap \Omega = \emptyset\) or \(N^0 \cap \Omega = \Omega\). In other words \(W\) is imprimitive if transitive and there are blocks. Generally for \((Z_n, +)\) on \(\{0, 1, 2, \ldots, n - 1\}\), if \(k \mid n\) and \(r \in \Omega\) then \(\{\ell \in \Omega; \ell = r \mod k\}\) is a block. Example, taking \(n = 6\), then \(\{\ell \in \Omega; \ell = 2 \mod 3\} = \{2, 5\}\) is a block.

Theorem 2.13: The centre of \(W\) written \(Z(w)\) is defined where \(k = \{beB \setminus \delta b = b \forall \delta \in \Delta\}\) by \(Z(w) = fb \setminus (fb)(f_1, b_1) = (f_1, b_2)(fb); \forall f_1 \in G, b_1 \in B\). For \(f b \in Z(w)\) if and only if \(f f_1^{b^{-1}} b_1 = f f_2^{b^{-1}} b_2 \forall f_1, b \in G, b \in B\) ..............(1.1)

Proof:
Substituting \(b_1 = 1\), (1.1) becomes \(f f_1^{b^{-1}} b_1 = f f_1^{b^{-1}} b_2 \forall f_1 \in G\) ..............(1.2)
Putting \(f_1 = 1\) in (1.1) we get \(f b = f f_2^{b^{-1}} b_2 \forall f_1 \in G\) ..............(1.3)
Therefore, for \(f b \in Z(w)\), then \(Z(B)\).

Claim:
If \(A \neq 1, f b \in Z(w)\) and \(b \in Z(B)\), then \(\delta b = \delta\) for all \(\delta \in \Delta\) ..............(1.4)
Choose \(f_1 \in G\) such that \(f_1(\delta) = c \neq 1, a \in A \neq 1\) and \(f_1(\delta') = 1, \forall \delta' \neq \delta\) ..............(1.5)
And so, from (1.2) we have \(f_1(\delta)f(\delta) = f(\delta)f_1(\delta b) = f(\delta), \forall \delta b \neq \delta\) showing that \(f_1(\delta) = 1\) is a contradiction from equation (1.5). Thus \(\delta b = \delta\) for all \(\delta \in \Delta\). Hence our claim. But (1.2) also implies that for all \(\delta \in \Delta, f_1(\delta) f(\delta) = f(\delta)f_1(\delta b) = f(\delta)f_1(\delta)\). So \(f(\delta) \in Z(A)\), for all \(\delta \in \Delta\) ..............(1.6)
But (1.3) implies that \(f_1(\delta b_1) = f(\delta) \forall b_1 \in B\) ..............(1.7)
Since \(f\) is constant over orbits of \(B\) in \(\Delta\) and from (1.4), (1.6) and (1.7) we conclude that for \(A \neq 1, f b \in Z(w)\) if and only if

\[ Z(w) = \begin{cases} Z(B) & \text{if } A \neq 1 \\ \{\pi Z(B) \cap \Omega\} & \text{otherwise} \end{cases} \]

3. RESULTS AND DISCUSSION

Example: Let the permutation group \(A = \{(1), (1 2 4), (1 4 2)\}\) and \(B = \{(1), (5 6)\}\) act on the sets \(\Gamma = \{1, 2, 4\}\) and \(\Delta = \{5, 6\}\) respectively. Also let \(G = \Lambda^\Delta = \{f \setminus \Delta \rightarrow A\}\), then \(|P| = |A|^{|\Delta|} = 3^2 = 9\). The mappings are as follows:
\[
\begin{align*}
f_1: 5 &\rightarrow (1), \quad 6 \rightarrow (1) \\
f_2: 5 &\rightarrow (1 2 4), \quad 6 \rightarrow (1 2 4) \\
f_3: 5 &\rightarrow (1 4 2), \quad 6 \rightarrow (1 4 2) \\
f_4: 5 &\rightarrow (1), \quad 6 \rightarrow (1 2 4) \\
f_5: 5 &\rightarrow (1 2 4), \quad 6 \rightarrow (1) \\
f_6: 5 &\rightarrow (1 2 4), \quad 6 \rightarrow (1) \\
f_7: 5 &\rightarrow (1 4 2), \quad 6 \rightarrow (1) \\
f_8: 5 &\rightarrow (1 2 4), \quad 6 \rightarrow (1 4 2) 
\end{align*}
\]
With respect to the operators \((f_{1}b)_{2} = (f_{1}b^{-1})_{2}\) it can be verified that \(G\) is a group for \(\delta \epsilon \Delta\). We define the action of \(B\) on \(G\) as \(f^{b}(\delta) = f'(\delta)_{2}\). \(\forall \ g \in b \in B\), \(\delta \epsilon \Delta\), then \(B\) acts on \(G\) as groups.

Define \(W = A \wr B\), the semi-direct product of \(G\) by \(B\) in that order; that \(W = \{ f, b : f \in G, b \in B \}\) so that \(W\) is a group with respect to the operation \((f_{1}b_{1})(f_{2}b_{2}) = (f_{1}b_{1}^{-1})(b_{1}b_{2})\) so that our \(b_{1} = (1)\) and \(b_{2} = (4,5)\). Thus this will give the element of \(W\) as
\[
(f_{1}b_{1}), (f_{2}b_{1}), (f_{3}b_{1}), (f_{4}b_{1}), (f_{5}b_{1}), (f_{6}b_{1}), (f_{7}b_{1}), (f_{8}b_{1}), (f_{9}b_{1}), (f_{10}b_{1}), (f_{11}b_{2}), (f_{12}b_{2}), (f_{13}b_{2}) \cdot
\]
Next we define our \(W\) on \(\Gamma \times \Delta\) as shown below: \((\alpha) f b = (\alpha f(\delta), \delta b)\). But also \(\Gamma \times \Delta = \{(1,5), (1,6), (2,5), (2,6), (4,5), (4,6), (4,5)\}.

And the following permutations by the action of \(W\) on \(\Gamma \times \Delta\)

\[
(\Gamma \times \Delta)f_{1}b_{1} = \begin{pmatrix}
(1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\
(1,5)(1,6)(2,5)(2,6)(4,5)(4,6)
\end{pmatrix}
\]

\[
(\Gamma \times \Delta)f_{2}b_{1} = \begin{pmatrix}
(1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\
(2,5)(2,6)(4,5)(4,6)(1,6)(1,6)
\end{pmatrix}
\]

\[
(\Gamma \times \Delta)f_{3}b_{1} = \begin{pmatrix}
(1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\
(4,5)(4,6)(1,5)(1,6)(2,5)(2,6)
\end{pmatrix}
\]

\[
(\Gamma \times \Delta)f_{4}b_{1} = \begin{pmatrix}
(1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\
(1,5)(2,6)(2,5)(4,6)(4,5)(1,6)
\end{pmatrix}
\]

\[
(\Gamma \times \Delta)f_{5}b_{1} = \begin{pmatrix}
(1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\
(1,5)(4,6)(2,5)(1,6)(4,5)(6,4)
\end{pmatrix}
\]

\[
(\Gamma \times \Delta)f_{6}b_{1} = \begin{pmatrix}
(1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\
(2,5)(1,6)(4,5)(2,6)(1,5)(4,6)
\end{pmatrix}
\]

\[
(\Gamma \times \Delta)f_{7}b_{1} = \begin{pmatrix}
(1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\
(4,5)(1,6)(2,5)(2,6)(1,5)(4,6)
\end{pmatrix}
\]

\[
(\Gamma \times \Delta)f_{8}b_{1} = \begin{pmatrix}
(1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\
(4,5)(2,6)(1,5)(4,6)(2,5)(1,6)
\end{pmatrix}
\]

\[
(\Gamma \times \Delta)f_{9}b_{2} = \begin{pmatrix}
(1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\
(1,6)(1,5)(2,6)(2,5)(4,6)(1,5)
\end{pmatrix}
\]

\[
(\Gamma \times \Delta)f_{10}b_{2} = \begin{pmatrix}
(1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\
(2,5)(2,5)(4,6)(1,6)(1,5)(2,4)
\end{pmatrix}
\]

\[
(\Gamma \times \Delta)f_{11}b_{2} = \begin{pmatrix}
(1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\
(4,6)(4,5)(1,6)(1,5)(2,6)(1,5)
\end{pmatrix}
\]

\[
(\Gamma \times \Delta)f_{12}b_{2} = \begin{pmatrix}
(1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\
(4,5)(1,6)(2,5)(2,6)(4,5)(6,1)
\end{pmatrix}
\]

\[
(\Gamma \times \Delta)f_{13}b_{2} = \begin{pmatrix}
(1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\
(1,6)(2,5)(2,6)(4,5)(6,1)(1,5)
\end{pmatrix}
\]

\[
(\Gamma \times \Delta)f_{14}b_{2} = \begin{pmatrix}
(1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\
(1,6)(4,5)(2,6)(1,5)(4,6)(2,5)
\end{pmatrix}
\]

\[
(\Gamma \times \Delta)f_{15}b_{2} = \begin{pmatrix}
(1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\
(2,6)(1,5)(4,6)(2,5)(1,6)(4,5)
\end{pmatrix}
\]
\[(\Gamma \times \Delta)f_{ab_2} = \begin{pmatrix} (1,5)(1,6)(2,5)(2,6)(4,5)(4,6) \\ (2,6)(4,5)(4,6)(1,5)(1,6)(2,5) \end{pmatrix} \]

If we rename the symbols for convenience as \((1,5) = 1, (1,6) = 2, (2,5) = 3, (2,6) = 4, (4,5) = 5, (4,6) = 6\). And so in cyclic form, we write these permutations as: \((1), (1\ 3\ 5)(2\ 4\ 6), (1\ 5\ 3)(2\ 6\ 4), (2\ 4\ 6), (1\ 3\ 5), (1\ 5\ 3)(2\ 6\ 4), (1\ 5\ 3)(2\ 4\ 6), (1\ 2)(3\ 4)(5\ 6), (1\ 4\ 5\ 2\ 3\ 6), (1\ 6\ 3\ 2\ 5\ 4), (1\ 2\ 3\ 4\ 5\ 6), (1\ 6\ 5\ 4\ 3\ 2), (1\ 2\ 5\ 6\ 3\ 4), (1\ 4\ 3\ 6\ 5\ 2), (1\ 4)(2\ 5)(3\ 6)\)

\[\text{and}\quad (1\ 6)(2\ 3)(4\ 5)\]

3. CONCLUSION

The research paper studied and presented the conditions in which the wreath products of permutation groups prove to be faithful, transitive and primitive. An example was presented and used as an illustration to support the findings.

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