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The Interest Rate Algebra

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Abstract

This article considers the structure of interest rate, applied for discounting of risky cash flows. The purpose of the article is a presentation of ways of reflecting inflation and risks in the calculation of risk discount rate. In introduction on base of well-known dependences is shown that risk premium depends on inflation rate and (for multiplicative-type models) risk free rate. In the first part of the article three Interest rate algebras are presented. They describe the attitude between nominal discount rate, risk free rate, inflation rate and risk-premium. This algebras can presents in additive-type or multiplicative-type versions and have given risk premium value without detailed description the structure of risk premium. The second part of the paper has more detailed attitude between risk premium, risk free rate and mathematical expectation of losses because of bankruptcy/default. It is shown that the obtained dependences are slightly different and depends on the initial preconditions calculation: the principle of arbitration or the principle of certainly equivalent.
Key words: Interest rate, risk free rate, risk premium, probability of losses

1. Introduction

There are many different interpretations for structure of interest rate: unbiased expectations theory, liquidity preference theory, liquidity premium. Also exist many different models for discount rate valuation, such as CAPM, MCAPM, APM and etc. However most of this models don't give possibility of direct connection between such parameters as inflation, real rate and risk premium. This article contains a version of interest rate algebra sets developed by the author and applicable for the interpretation of structure for risk-free (in first part) and risky (in second part) interest rates.

According to the established tradition, the analysis of the structure of interest rate for the purpose of its subsequent use in discounting exercises involves the following relations:

Irving Fisher Expression [1]:

$$r = (1 + r_r) \cdot (1 + i) - 1, \qquad (1)$$

where, r – nominal rate of interest,
r_r – real rate of interest,
i – the rate of inflation for the period,

Furthermore, decomposition of an interest rate into a risk-free component and the risk premium can be expressed as [2], [3], [4], [5]:

$$r = r_f + pr, \qquad (2)$$

where, r – the interest rate specific to future cash flows from a project (asset) with a certain investment risk,

r_f – interest rate on risk-free investments,

pr – risk premium for bearing the risk of investing into similar projects (assets).

In turn, the risk-free rate can be represented in an additive form (following the concept of simple percents):

$$r_f = r_{fr} + i, \quad (3)$$

where, r_{fr} – real interest rate on risk-free investments,

or in a multiplicative form (following the concept of compound interest):

$$r_{fm} = (1 + r_{fm})(1 + i) - 1, \quad (4)$$

where, r_{fm} – nominal risk-free interest rate used for complex compound interest,

r_{frm} – real risk-free interest rate used for complex compound interest.

Thus, the work of the classics of investment theory affords the conclusion that the level of interest rate is a function of the risk-free rate (net of inflation), the rate of inflation, and the risk premium component:

$$r = r_{fr} + i + pr_{ra}, \quad (5)$$

$$r = (1 + r_{frm}) \cdot (1 + i) \cdot (1 + pr_{rm}) - 1, \quad (6)$$

where, r – is a nominal rate of interest applicable for discounting cash flows from a risky investment project (i.e. risky cash flows),

r_{fr} – real (net of inflation) risk-free rate of interest,

pr_{ra} – risk premium applicable in an additive-type model, as in (5), given a separate accounting for the inflationary component,

pr_{rm} - risk premium applicable in a multiplicative-type models, as in (6), given a separate accounting for the inflationary component,

r_{frm} – the value of the real risk-free rate used in multiplicative-type models, as in (6).

It is evident that there exist the following relations between pr_{ra} and pr_{rm} :

$$pr_{rm} = \frac{pr_{ra}}{1 + i + r_{frm}(1 + i)} = \frac{pr_{ra}}{1 + r_f}, \quad (7)$$

$$pr_{ra} = pr_m(1 + r_f), \quad (8)$$

where, r_f – the value of the nominal risk-free rate.

If we decompose the real interest rate r_r in expression (1) into the risk-free real rate and the (net of inflation) risk premium (pr^{extra}_r)[1] ($r_r = (1 + r_{fr})(1 + pr^{extra}_{rm}) - 1$), the following transformation of the expression obtains:

$$\begin{aligned} r &= (1 + r_{frm})(1 + pr^{extra}_{rm})(1 + i) - 1 = \\ &= r_{frm} + i + ir_{frm} + r_{frm}pr^{extra}_{rm} + pr^{extra}_{rm} + \\ &+ ipr^{extra}_{rm} + ir_{frm}pr^{extra}_{rm} = \\ &= r_f + pr^{extra}_{mr}(1 + i)(1 + r_{frm}). \end{aligned} \quad (1a)$$

The last components in expression (1a) is the observed (ex ante) risk-premia for the respective (i.e. additive or multiplicative) models. These expressions show that the inflation exerts immediate influence on the ex-ante risk premium. Moreover, Expression (1a) indicates that the ex-ante risk premium is also affected by the value of the implied real risk-free rate.

[1] To avoid double-counting in practical terms, only that part of the risk premium (pr^{extra}_{ra}) is to be accounted for in this exercise which is not already implicitly assumed in the risk-free rate, since risk-free rate metrics used in practice, arguably, admit of the presence of a small element of risk in them.

1.1. Interest Rate Algebra -1 (IRA-1)

Expressions (1a) and (1b) demonstrate that the detailed study of the subject of interest rates leads to realization that the number of structural representations for interest rates is not exhausted merely by the expressions (1)-(4). This is due to the following factors:

- A risk-free rate can be represented either in nominal (r_f) or real (r_{fr}) terms;
- A risk-free rate may either include some residual risk elements (r_f and r_{fr}), or be netted of any such elements (r_{nf} and r_{nfr});
- A risk-free rate can be determined by the rule of simple (r_f, r_{nf}, r_{nfr}) and compound interest (r_{fm}, r_{nfm}, r_{nfrm});
- The risk premium can be represented differently for the additive (pr_a) or multiplicative (pr_m) interest rate models;
- The risk premium can be represented in nominal (pr) or real (pr_r) terms;
- The risk premium can be used in conjunction with "risk-free rates" which incorporate some residual risk elements - r_f, r_{fr} (in this case the risk premium shall be denoted with the "extra" superscript pr^{extra}), as well as in conjunction with absolute ("pure") risk-free rates - r_{nf}, r_{nfr} (in this case, the notation for such risk premia does without the «extra» superscript - pr).

Having regard to these circumstances, an attempt is made below to list all possible detailed specifications of the interest rate structure, which can be employed in the context of discounting for risky cash flows (in situations where the risk factor is incorporated in the discount rates, and not through adjusting the cash flows themselves, as the case may be):

$$r = r_{nfr} + i + pr_{ra}, \qquad (9)$$

$$r = r_{nfr} + pr_a, \qquad (10)$$

$$r = r_{nf} + pr_{ra}, \qquad (11)$$

$$r = r_{fr} + pr_a^{extra}, \qquad (12)$$

$$r = r_{fr} + i + pr_{ra}^{extra}, \qquad (13)$$

$$r = r_f + pr_{ra}^{extra}, \qquad (14)$$

$$r = (1+r_{nfm})(1+pr_{rm}) - 1, \qquad (15)$$

$$r = (1+r_{nfrm})(1+pr_{rm})(1+i) - 1, \qquad (16)$$

$$r = (1+r_{fm})(1+pr_{rm}^{extra}) - 1, \qquad (17)$$

$$r = (1+r_{frm})(1+pr_{mr}^{extra})(1+i) - 1, \qquad (18)$$

$$r = (1+r_{nfrm})(1+pr_m) - 1, \qquad (19)$$

$$r = (1+r_{frm})(1+pr_m^{extra}) - 1, \qquad (20)$$

$$r = r_{fr} + \alpha \cdot i + (1-\alpha) \cdot i + pr_{ra}^{extra} = r_{fpr} + pr_{pra}^{extra}, \qquad (21)$$

$$r = r_{nfr} + \alpha \cdot i + (1-\alpha) \cdot i + pr_{ra} = r_{nfpr} + pr_{pra}, \qquad (22)$$

where:

r – the nominal rate of interest/return that accounts for risks,

r_{nfr} - pure real risk-free rate (i.e. one net of all risks and inflation),

r_{nf} – pure nominal risk-free rate,

i – the rate of inflation,

r_f – nominal risk-free rate, that includes some residual elements of risk in practical terms (residual risk elements),

r_{fr} – real risk-free rate with some residual risk elements,

r_{nfm} – pure nominal risk-free rate used in multiplicative-type models,

r_{nfrm} – pure real risk-free rate used in multiplicative-type models,

r_{fm} – nominal risk-free rate used in multiplicative-type models, that includes some residual elements of risk in practical terms (residual risk elements),

r_{frm} – real risk-free rate used in multiplicative-type models, that includes some residual elements of risk in practical terms (residual risk elements),

pr_{rm} – full risk premium in the multiplicative representation, net of inflation,

pr_m – full risk premium in the multiplicative representation, incorporating inflation,

pr_{ra} – full risk premium in the additive representation, net of inflation,

pr_a – full risk premium in the additive representation, incorporating inflation,

pr^{extra}_m – partial risk premium over and above the residual risk elements in the risk-free rate r_f, in the multiplicative representation (incorporating inflation),

pr^{extra}_{rm} – partial risk premium over and above the residual risk elements in the risk-free rate r_f, in the multiplicative representation (net of inflation),

pr^{extra}_a - partial risk premium over and above the residual risk elements in the risk-free rate r_f, in the additive representation (incorporating inflation),

pr^{extra}_{ra} – partial risk premium over and above the residual risk elements in the risk-free rate r_f, in the additive representation (net of inflation),

r_{fpr} – risk-free rate with partial account for inflation: $r_{fpr} = r_{fr} + \alpha i$,

r_{nfpr} – pure risk-free rate with partial account for inflation: $r_{nfpr} = r_{nfr} + \alpha i$,

pr^{extra}_{pra} – partial risk premium over and above the residual risk elements in the risk-free rate r_{fpr}, in the additive representation (with partial account for inflation): $pr^{extra}_{pra} = pr^{extra}_{ra} + (1 - \alpha)i$,

pr_{pra} – risk premium with a partial account for inflation applicable in the additive models and to be used in conjunction with the r_{nfpr} risk-free rate partially accounting for inflation: $pr_{pra} = pr_{ra} + (1 - \alpha)i$,

α – a share (fraction) of inflation reflected (included) in the risk-free rate, $0 \leq \alpha \leq 1$,

$(1 - \alpha)$ – the remaining share (fraction) of inflation reflected (included) in the risk premium.

It can be ascertained that Models in (21) and (22) reflect valid options for inflation accounting -- both in relation to the risk-free rate and the risk premium.

In addition to the basic expressions (9) - (22), reflecting the variety of designs for the calculation of risky discount rate of nominal cash flows also will show how the link between the individual components of the rates:

$$r_f = r_{fr} + i, \tag{23}$$

$$r_{nf} = r_{nfr} + i, \tag{24}$$

$$r_{fr} = r_{nfr} + \Delta_a, \tag{25}$$

$$pr_a = pr_{ra} + i, \tag{26}$$

$$\Delta_a \equiv pr_a - pr_a^{extra}, \tag{27}$$

$$\Delta_a \equiv pr_{ra} - pr_{ra}^{extra}, \tag{28}$$

$$pr_a^{extra} = pr_a - \Delta_a, \tag{29}$$

$$pr_{ra}^{extra} = pr_a^{extra} - i, \tag{30}$$

$$pr_{ra}^{extra} = pr_{ra} - \Delta_a, \tag{31}$$

$$r_{nfrm} = \frac{1 + r_{frm}}{1 + \Delta_m} - 1, \tag{32}$$

$$\Delta_m = \Delta_a \frac{r_{frm}}{r_{fr}}, \tag{33}$$

$$r_{nfm} = (1 + r_{nfrm})(1 + i) - 1, \tag{34}$$

$$pr_m = (1 + pr_m^{extra})(1 + \Delta_m) - 1, \quad (35)$$

$$pr_m = (1 + pr_{mr}^{extra})(1 + i)(1 + \Delta_m) - 1, \quad (36)$$

$$pr_m^{extra} = (1 + pr_{mr}^{extra})(1 + i) - 1, \quad (37)$$

$$pr_{mr} = \frac{1 + pr_m}{1 + i} - 1, \quad (38)$$

$$pr_{mr}^{extra} = \frac{1 + r}{1 + r_{fm}} - 1, \quad (39)$$

$$pr_{mr}^{extra} = \frac{1 + pr_m^{extra}}{1 + i} - 1, \quad (40)$$

where, Δ_a - part of the risk premium, which is integrated into the traditional (used) risk-free rate used in additive models,

Δ_m - part of the risk premium, which is integrated into the traditional (used) risk-free rate used in multiplicative-type models.

1.2. Interest Rrate Algebra -2 (IRA-2)

As the dealings in actual professional practice are for the most part limited to the observed "risk-free" rates which contain (or may contain) the admixtures of risk elements (in other words, we have no data on the values of pure risk-free market interest rates r_{nfr} and r_{nf}), practical value attaches only to those Expressions, out of the set of Expressions (9)-(22), which do not contain pure risk-free rates (r_{nfr} and r_{nf}), i.e. to Expressions (12)-(14), (17)-(18), (20)-(21). Since all these Expressions carry the "extra" superscript in the notation for risk premia, it now makes sense to do away with using this superscript for sheer practicality. To avoid confusion in what follows, let us make use of a new notation, removing the "extra" superscript and replacing lower-case letters with the capital ones (it is possible, of course, to continue with using the lower-case letters -- bearing in mind that the applicable value of the risk premium is only partial, as some of its elements have actually been "woven" into the practically observed equivalent for the risk-free rate):

$$R = r_{fr} + i + \mathrm{Pr}_{ra}, \quad (41)$$

$$R = r_f + \mathrm{Pr}_{ra}, \quad (42)$$

$$R = r_{fr} + \mathrm{Pr}_a, \quad (43)$$

$$R = (1 + r_f)(1 + \mathrm{Pr}_{rm}) - 1, \quad (44)$$

$$R = (1 + r_{frm})(1 + \mathrm{Pr}_{rm})(1 + i) - 1, \quad (45)$$

$$R = (1 + r_{frm})(1 + \mathrm{Pr}_m) - 1, \quad (46)$$

$$R = r_{fpr} + \mathrm{Pr}_{pra}, \quad (47)$$

where, R is an equivalent of r , Pr – of pr_a^{extra} , and Pr_{pra} – of pr_{pra}^{extra} in terms of notation previously employed in the Expressions (9)-(22).

As seen from the above expressions, IRA-2 assumes a complete absence of risk elements in the risk-free rate. This option algebra, as well as IRA-1, allows for the possibility of presentation of models for valuing the discount rate in multiplicative and additive forms.

1.3. Interest Rate Algebra -3 (IRA-3)

Having regard to the fact that appraisers and investment analysts for the most part limit themselves to the consideration of additive-type interest rate models, the immediately preceding algebra of interest rates (IRA-2) can be further simplified by excluding from it all the multiplicative-type models and leaving in only the additive models. Simultaneously, with the multiplicative models no longer featuring in the algebra, we shall exclude from the ensuing expressions all "a" subscripts denoting the membership in the additive-type model class. As a result, this new, simplest, algebra set features only additive models in the following representations:

$$r = r_{fr} + i + pr_r\,, \tag{48}$$

$$r = r_f + pr_r\,, \tag{49}$$

$$r = r_{fr} + pr, \tag{50}$$

$$r = r_{fr} + \alpha i + (1-\alpha)i + pr_r = = r_{fpr} + pr_{pr}\,, \tag{51}$$

where r is the equivalent of R, pr_r - of Pr_{ra}, pr - of Pr_{ra}, pr_{pr} - of Pr_{pra} in terms of notation previously used in the Expressions (41) – (43), (47), namely:

r – is the nominal rate of interest applicable for discounting after-tax cash flows from a risky investment project (risky cash flows),

r_{fr} – real (i.e. net of inflation) risk-free interest rate (essentially reflecting the above mentioned "usurious" (i.e. "net-net-net") interest),

pr_r – real (net of inflation) risk premium,

pr – nominal risk premium that includes the inflationary component,

ά – a share (fraction) of inflation included (reflected) in the risk-free rate r_{fpr},

(1 - ά) – the remaining share (fraction) of inflation included (reflected) in the risk premium pr_{pr}.

As seen from the above expressions, in the General case, inflation can be taken into account as the risk-free rate and the risk premium. However, it is important to avoid double counting: in other words, inflation can only redistribute between the risk-free rate and a risk premium.

2. Accounting for Default and Insolvency Risks (IRA-4)

Let us now take up the subject of the impact of risk on discount rates from a different standpoint. The lack-of-arbitrage-opportunities condition can be expressed as follows:

- in additive form:

$$r_1 \cdot (1 - p_d) + (-k) \cdot p_d = r_f\,, \tag{52}$$

- in multiplicative form:

$$[1 + r_1(1 - p_d)] \times [1 + (-k)p_d] = 1 + r_{fm}, \tag{53}$$

where, p_d – probability of insolvency/default (or, of a shortfall in payments, put simply), k – losses given default (as a fraction of the amount outstanding),

r_f – risk-free rate used in additive-type models,

r_{fm} – risk-free rate used in multiplicative-type models,

r_1 – expected return on investment into shares, at a favorable outcome.

On the basis of Expressions (52), (53) it is possible to obtain an expressions linking the above-mentioned expected return with the risk-free rate and the parameters of risk:

$$r_1 = \frac{r_f + p_d \cdot k}{1 - p_d}. \tag{54}$$

$$r_{1m} = \frac{r_{fm} + p_d \times k}{1 - p_d(1 + k - kp_d)}. \tag{55}$$

Expressions (54)-(55), in turn, is amenable for the quantification of the risk premium:

-given the additive specification for the risk-free rate and the risk premium:

$$pr_{a1} = r_1 - r_f = = \frac{r_f + p_d \cdot k}{1 - p_d} - r_{fa} = = \frac{p_d(k + r_f)}{1 - p_d}\,, \tag{56}$$

- given the multiplicative specification for the risk-free rate and the risk premium:

$$pr_{m1} = \frac{1 + r_{1m}}{1 + r_{fm}} - 1 =$$
$$= \frac{p_d \times (k + r_{fm}(1 + k - kp_d))}{(1 - p_d(1 + k - kp_d)(1 + r_{fm})}. \quad (57)$$

On the other hand, the relation between the risky rate and risk parameters can be obtained from a different consideration:

$$\frac{1}{1 + r} = \frac{1 - p_d \cdot k}{1 + r_f}, \quad (58)$$

where, the numerator in the right-hand side of this equation reflects the adjustment to expected cash flows that transforms them into their certainty equivalents. Solving Equation (58) for r results in the following expression for the risky rate:

$$r_2 = \frac{r_f + p_d \cdot k}{1 - p_d \cdot k}, \quad (59)$$

where r_2 – is the estimate for risky rate obtained on the basis of Condition (58).

Expression (59) also allows for deriving an expression for the risk premium:

- for the additive relation between the risk-free rate and the risk premium:

$$pr_{a2} = r_2 - r_f =$$
$$= \frac{r_f + p_d \cdot k}{1 - p_d \cdot k} - r_f = \quad (60)$$
$$= \frac{p_d \cdot k \cdot (1 + r_f)}{1 - p_d \cdot k},$$

- for the multiplicative relation between the risk-free rate and the risk premium:

$$pr_{m2} = \frac{1 + r_2}{1 + r_f} - 1 = \frac{p_d \cdot k}{1 - p_d \cdot k}. \quad (61)$$

Despite their similarity, the Expressions (54) and (59) are not identical. The author of this Paper is hard put to give a conclusive explanation to the disparity between the formulas, however, it can be assumed that its nature is associated with the fact that in real life the number of possible event scenarios is significantly above the two outcomes implied in the initial lack-of-arbitrage hypothesis.

In this regard, and as of the writing, there is a reason to repose greater trust in the Expression (59).

Subsequent to deriving these formulas, the author of this Paper found in the literature[6][2] an expression similar (even to the point of notation) to one in (54), which, as suggested in the source, can be used for estimating returns on risky bonds. The respective passage from the work of W. Sharpe is reproduced below:

"*How high should a default risk premium be for a bond? According to one model [7], the answer depends both on the probability of default and on the possible financial losses of bondholders given the default. Consider a bond whose probability of default is constant every year (provided that the payments for previous years have been met). Let the probability of default during a given year be denoted as p_d. Assume that, if the payments are remiss on the bond, the owner of each bond recovers a part, equal (1 - λ), of its market price in effect a year ago. According to this model, the bond shall be fairly priced, if its yield to maturity, "y", equals to:*

[2] In W. Sharpe "Investments" (Russian edition, by Infra-M Publishers, Moscow, 2007). pp. 432-433, at the point where the work of Gordon Pye (*Gordon Pye «Gauging the Default Premium», Financial Analysts Journal, 30, no.1 (January/February 1974), pp. 423-434)* is being referenced.

$$y = \frac{\overline{y} + \lambda p_d}{1 - p_d}, \qquad (15.4[3]) \qquad (62)$$

where, y denotes the expected yield to maturity of the bond. The difference, d, between the expected yield to maturity "y" and the expected [baseline] yield $\overline{y}$ has been previously alluded to as the default risk premium. Using the expression (15.4), we can see that this difference for fairly priced bonds should be equal to:

$$d = y - \overline{y} = \left(\frac{\overline{y} + \lambda p_d}{1 - p_d}\right) - \overline{y}. \qquad (15.5)\text{ "} \qquad (63)$$

(end of the quote).

In conclusion, it bears mentioning that cash flow discounting can be exercised in one of the two following ways:

- Each period payment is discounted using a discount rate specific to that period;
- A single discount rate is used for all periods - which corresponds to the "duration" of expected cash flows. The problem encountered with this approach is the difficulty of giving simultaneous/summary estimation to a set of rates in the form of an average expected interest rate (scoping over the investment horizon) – estimation which would not fail to reflect all the possible risks in the capital market.

3. Conclusions

On a final note, it is necessary to sum up the principal point covered in this paper.

- First of all, separation of components of risky interest rate on risk-free rate and risk premium can be done in a relatively correct ways and still plenty of wrong ways. Perhaps this is why so often errors in assessing the values of the discount rate, when often overlooked factors, or are counted twice.
- Secondly, the risk premium applied when calculating the nominal discount rates, in addition to risk factors, also depends on the level of risk-free interest rates and inflation (see (1a), (54), (55), (59)).
- Thirdly, the value of the risk premium has a nonlinear dependence of the expected losses/damages/bankruptcy or loss of the shortfall in expected revenues (see (56), (57), (60), (61)).
- Fourthly, model estimates of the risk premium should correspond to the used model of the calculation of the discount rate (see (54)-(57, (59)-(61)).
- Finally, taking into account the existing «backlash» in the values of the nominal risky interest rates under equal values of the input parameters of the model (because of the differences in models (54), (55), (59)), it should be acknowledged that at the time of writing, the minimum error values calculation risky rate is from 5% to 10% (in relative percentages).

Received in this article the results can hope to reach a more accurate calculations and the avoidance of errors in estimating discount rates.

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Jan Mossin, "Equilibrium in a Capital Asset Market", Econometrica, 34, no.4 (October 1966), pp.768-783.

[3] For the integrity of exposition, the numbering of formulas in the continuing quotation below is maintained in a double style—the original one (to the left), and one in conformity with the ongoing sequence in the present work (to the right).

William F. Sharpe, Gordon J.Alexander, Jeffery V. Bailey "Investments", 5th ed., Prentice Hall International, Inc., USA, 1995 (Russian edition, by Infra-M Publishers, Moscow, 1997, pp. 432-433).
Gordon Pye, «Gauging the Default Premium», Financial Analysts Journal, 30, no.1 (January/February 1974), pp. 423-434.

4.Appendix

4.1. Examples of Calculations with the use of Algebras of Interest Rates

In this Annex consider the examples of calculations using the above expressions. For easy comparison of the text of the application will be most closely correspond to the above text main part of the article.

Let's begin with the expression I. Fischer. Let the value of the real risk-free rate is 7.27%, inflation rate - 10%. Then, in accordance with (1) we obtain the value of the nominal interest rate:

$$r = (1+r_r)\cdot(1+i)-1 =$$
$$(1+0.0727)(1+0.1)-1 = 0.18. \quad (1A)$$

Now let assume that the nominal risk-free rate is 13%, and a risk premium of 5%. Then, in accordance with (2) we get:

$$r = r_f + pr =$$
$$= 0.13+0.05 = 0.18. \quad (2A)$$

Now suppose that the actual (cleared from inflation) risk-free rates used in a simple compound interest is 3%. Then, in accordance with (3) the value of the nominal risk-free rate, calculated in the framework of the additive model (following the concept of simple interest) will be as follows:

$$r_f = r_{fr} + i =$$
$$= 0.03+0.1 = 0.13. \quad (3A)$$

Let the value of real (cleared from inflation) risk-free rates used for complex compound the interest is 2.7%. Then, in accordance with (4) the value of the nominal risk-free rate, calculated in the framework of the multiplicative model (following the concept of compound interest) will be as follows[4]:

$$r_{fm} = (1+r_{frm})(1+i)-1 = 1.027\times 1.1-1 = 0.13. \quad (4A)$$

Thus, from the works of the classics, you can conclude that the level of interest rates is a function of the cleared from inflation risk-free rate, inflation and premium for the risk:

$$r = r_{fr} + i + pr_{ra} = 0.03+0.1+0.05 = 0.18, \quad (5A)$$

$$r = (1+r_{frm})\times(1+i)\times(1+pr_{rm})-1 =$$

$$r = (1+r_{frm})\times(1+i)\times(1+pr_{rm})-1 =$$
$$= 1.027\times 1.1\times 1.044-1 = 0.18, \quad (6A)$$

where the value of the pr_{rm} - risk premiums applied when you use the multiplicative model of type (6A) with a separate accounting of the inflationary component is specifically calculated.

It is obvious that between the pr_{ra} and pr_{rm} there are the following dependencies:

$$pr_{rm} = \frac{pr_{ra}}{1+i+r_{frm}(1+i)} = \frac{pr_{ra}}{(1+r_f)} =$$

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[4] In the example we specially selected parameter values r_{fr} and r_{frm} so that the calculated over one year interest are equal (i.e., $r_f = r_{fm}$), implying capitalization of interest once at the end of the year.

$$= \frac{0{,}05}{1+0{,}1+0{,}027(1+0{,}1)} = \frac{0{,}05}{1+0{,}13} = 0.044, \qquad (7A)$$

$$pr_{ra} = pr_{rm}(r_{frm}(1+i)+i+1) = pr_{rm}(1+r_f\ ) =$$
$$= 0.044\times(1+0.13) = 0.05. \qquad (8A)$$

(up to the rounding)

If we decompose the real interest rate r_r in expression (1) into the risk-free real rate and the (net of inflation) risk premium (pr^{extra}_r)[5] ($r_r = (1 + r_{fr})(1 + pr^{extra}_{rm}) - 1$), the following transformation of the expression obtains:

$$r = (1+r_{frm})(1+pr^{extra}_{rm})(1+i)-1 =$$
$$= r_{frm} + i + ir_{frm} + r_{frm}pr^{extra}_{rm} +$$
$$+ pr^{extra}_{rm} + ipr^{extra}_{rm} + ir_{frm}pr^{extra}_{rm} =$$
$$= r_f + pr^{extra}_{mr}(1+i)(1+r_{frm}) =$$
$$= 0.13+0.044\times1.1\times1.027 = 0.18. \qquad (1aA)$$

4.2. Interest Rate Algebra -1 (IRA-1)

Consider the following calculations with the use of IRA-1. The parameter values used will correspond to the previously adopted, or (for the new parameters) will be used by other values. The order of values of the parameters will strictly correspond to the sequence of model parameters:

$$r = r_{nfr} + i + pr_{ra} = 0.01+0.1+0.07 = 0.18, \text{(9A)}$$

$$r = r_{nfr} + pr_a = 0.01+0.17 = 0.18, \text{(10A)}$$

$$r = r_{nf} + pr_{ra} = 0.11+0.07 = 0.18, \text{(11A)} \qquad \text{(11П)}$$

$$r = r_{fr} + pr^{extra}_{a} = 0.03+0.15 = 0.18, \text{(12A)} \qquad \text{(12П)}$$

$$r = r_{fr} + i + pr^{extra}_{ra} = \\ = 0.03+0.1+0.05 = 0.18, \qquad \text{(13A)} \qquad \text{(13П)}$$

$$r = r_f + pr^{extra}_{ra} = \\ = 0.13+0.05 = 0.18, \qquad \text{(14A)} \qquad \text{(14П)}$$

$$r = (1+r_{nfm})(1+pr_{rm})-1 = \\ = 1.11\times1.063\text{-}1 = 0.18, \qquad \text{(15A)} \qquad \text{(15П)}$$

$$r = (1+r_{nfrm})(1+pr_{rm})(1+i)-1 = \\ = 1.009\times1.063\times1.1\text{-}1 = 0.18, \qquad \text{(16A)} \qquad \text{(16П)}$$

$$r = (1+r_{fm})(1+pr^{extra}_{rm})-1 = \\ = 1.13\times1.044\text{-}1 = 0.18, \qquad \text{(17A)} \qquad \text{(17П)}$$

$$r = (1+r_{frm})(1+pr^{extra}_{mr})(1+i)-1 = \\ = 1.027\times1.044\times1.1\text{-}1 = 0.18, \qquad \text{(18A)} \qquad \text{(18П)}$$

[5] To avoid double-counting in practical terms, only that part of the risk premium (pr^{extra}_{ra}) is to be accounted for in this exercise which is not already implicitly assumed in the risk-free rate, since risk-free rate metrics used in practice, arguably, admit of the presence of a small element of risk in them.

$$r = (1 + r_{nfrm})(1 + pr_m) - 1 = = 1.009 \times 1.169 - 1 = 0.18, \quad (19A) \quad (19\Pi)$$

$$r = (1 + r_{frm})(1 + pr_m^{extra}) - 1 = = 1.027 \times 1.148 - 1 = 0.18, \quad (20A) \quad (20\Pi)$$

$$r = r_{fr} + \alpha \cdot i + (1 - \alpha) \cdot i + pr_{ra}^{extra} = r_{fpr} + pr_{pra}^{extra} = 0.03 + 0.7 \times 0.1 + + (1 - 0.7) \times 0.1 + 0.07 = 0.1 + 0.08 = 0.18, \quad (21A) \quad (21\Pi)$$

$$r = r_{nfr} + \alpha \cdot i + (1 - \alpha) \cdot i + + pr_{ra} = r_{nfpr} + pr_{pra} = = 0.01 + 0.7 \times 0.1 + + (1 - 0{,}7) \times 0.1 + 0.07 = = 0.08 + 0.1 = 0.18. \quad (22A) \quad (22\Pi)$$

In addition to the basic expressions (9) - (22), reflecting the variety of designs for the calculation of risky discount rate for nominal cash flows also will show how the link between the individual components of the rates is exist:

$$r_f = r_{fr} + i = 0.03 + 0.1 = 0.13, \quad (23A)$$

$$r_{nf} = r_{nfr} + i = 0.01 + 0.1 = 0.11, \quad (24A)$$

$$r_{fr} = r_{nfr} + \Delta_a = 0.01 + 0.02 = 0.03, \quad (25A)$$

$$pr_a = pr_{ra} + i = 0.07 + 0.1 = 0.17, \quad (26A)$$

$$\Delta_a \equiv pr_a - pr_a^{extra} = 0.17 - 0.15 = 0.02, \quad (27A)$$

$$\Delta_a \equiv pr_{ra} - pr_{ra}^{extra} = 0.07 - 0.05 = 0.02, \quad (28A)$$

$$pr_a^{extra} = pr_a - \Delta_a = 0.17 - 0.02 = 0.15, \quad (29A)$$

$$pr_{ra}^{extra} = pr_a^{extra} - i = 0.15 - 0.1 = 0.05, \quad (30A)$$

$$pr_{ra}^{extra} = pr_{ra} - \Delta_a = 0.07 - 0.02 = 0.05, \quad (31A)$$

$$r_{nfrm} = \frac{1 + r_{frm}}{1 + \Delta_m} - 1 = \frac{1.027}{1.018} - 1 = 0.009, \quad (32A)$$

$$\Delta_m = \Delta_a \frac{r_{frm}}{r_{fr}} = 0.02 \times \frac{0.027}{0.03} = 0.018, \quad (33A)$$

$$r_{nfm} = (1 + r_{nfrm})(1 + i) - 1 = = 1.009 \times 1.11 - 1 = 0.11, \quad (34A)$$

$$pr_m = (1 + pr_m^{extra})(1 + \Delta_m) - 1 = = 1.1486 \times 1.018 - 1 = 0.169, \quad (35A)$$

$$pr_m = (1 + pr_{mr}^{extra})(1 + i)(1 + \Delta_m) - 1 = = 1.044 \times 1.1 \times 1.018 - 1 = 0.169, \quad (36A)$$

$$pr_m^{extra} = (1 + pr_{mr}^{extra})(1+i) - 1 = \\ = 1.044 \times 1.1 - 1 = 0.148, \quad (37A)$$

$$pr_{mr} = \frac{1+pr_m}{1+i} - 1 = \frac{1.169}{1.1} - 1 = 0.063, \quad (38A)$$

$$pr_{mr}^{extra} = \frac{1+r}{1+r_{fm}} - 1 = \frac{1.18}{1.13} - 1 = 0.044, \quad (39A)$$

$$pr_{mr}^{extra} = \frac{1+pr_m^{extra}}{1+i} - 1 = \frac{1.148}{1.1} - 1 = \\ = 0.044. \quad (40A)$$

4.3. Interest Rate Algebra -2 (IRA-2)

Consider the following calculations with the use of the IRA-2. The parameter values used will correspond to the previously adopted, or (for the new parameters) will be used by other values. The order of values of the parameters will strictly correspond to the sequence of model parameters:

$$R = r_{fr} + i + \Pr_{ra} = 0.03 + 0.1 + 0.05 = 0.18, \quad (41A)$$

$$R = r_f + \Pr_{ra} = 0.13 + 0.05 = 0.18, \quad (42A)$$

$$R = r_{fr} + \Pr_a = 0.03 + 0.15 = 0.18, \quad (43A)$$

$$R = (1 + r_{fm})(1 + \Pr_{rm}) - 1 = \\ = 1.13 \times 1.044 \text{-} 1 = 0.18, \quad (44A)$$

$$R = (1 + r_{frm})(1 + \Pr_{rm})(1+i) - 1 = \\ = 1.027 \times 1.044 \times 1.1 \text{-} 1 = 0.18, \quad (45A)$$

$$R = (1 + r_{frm})(1 + \Pr_m) - 1 = \\ = 1.027 \times 1.148 \text{-} 1 = 0.18, \quad (46A)$$

$$R = r_{fpr} + \Pr_{pra} = 0.1 + 0.08 = 0.18, \quad (47A)$$

4.4. Interest Rate Algebra -3 (IRA-3)

Consider the following calculations with the use of the IRA-3. The parameter values used will correspond to the previously adopted, or (for the new parameters) will be used by other values. The order of values of the parameters will strictly correspond to the sequence of model parameters:

$$r = r_{fr} + i + pr_r = \\ = 0.03 + 0.1 + 0.05 = 0.18, \quad (48A)$$

$$r = r_f + pr_r = \\ = 0.13 + 0.05 = 0.18, \quad (49A) \quad (49П)$$

$$r = r_{fr} + pr = \\ = 0.03 + 0.15 = 0.18, \quad (50A) \quad (50П)$$

$$r = r_{fr} + \alpha i + (1-\alpha)i + pr_r = r_{fpr} + pr_{pr} = 0.03 + 0.7 \times 0.1 + (1-0.7) \times 0.1 + 0.05 = = 0.1 + 0.08 = 0.18. \quad (51A)$$

(51П)

Assessment of Risk Premiums for Probable Bankruptcy, Defaults, Arrears (Interest Rates Algebra-4 (IRA-4))

Let the values of $p_d = 0.1$ and $k = 0.5$. Then from (52) and (53) can be obtained expressions connecting the specified expected return on risk-free rate and the risk variables:

$$r_1 = \frac{r_f + p_d \times k}{1 - p_d} = \frac{0.13 + 0.1 \times 0.5}{1 - 0.1} = 0.2, \quad (54A)$$

$$r_{1m} = \frac{r_{fm} + p_d \times k}{1 - p_d(1 + k - kp_d)} = \frac{0.13 + 0.1 \times 0.5}{1 - 0.1(1 + 0.5 - 0.5 \times 0.1)} = \quad (55A)$$
$$= 0.2105.$$

From (54) and (55), we can obtain expressions for the risk premiums:

- under the additive-type dependence between the risk-free rate and the risk premium:

$$pr_{a1} = r_1 - r_f = \frac{r_f + p_d \times k}{1 - p_d} - r_{fa} = \frac{p_d(k + r_f)}{1 - p_d} = \frac{0.1(0.5 + 0.13)}{1 - 0.1} = 0.07, \quad (56A)$$

- under the multiplicative-type dependence between the risk-free rate and the risk premium:

$$pr_{m1} = \frac{1 + r_{1m}}{1 + r_{fm}} - 1 = \frac{p_d \times (k + r_{fm}(1 + k - kp_d))}{(1 - p_d(1 + k - kp_d))(1 + r_{fm})} = \frac{0.1 \times (0.5 + 0.13(1 + 0.5 - 0.5 \times 0.1))}{(1 - 0.1(1 + 0.5 - 0.5 \times 0.1))(1 + 0.13)} = = 0.071.$$

(57A)

On the other hand, the relation between the risky rate and risk parameters can be obtained from a different consideration:

$$\frac{1}{1 + r} = \frac{1 - p_d \cdot k}{1 + r_f}, \quad (58)$$

where, the numerator in the right-hand side of this equation reflects the adjustment to expected cash flows that transforms them into their certainty equivalents. Solving Equation

(58) for r results in the following expression for the risky rate:

$$r_2 = \frac{r_f + p_d \cdot k}{1 - p_d \cdot k} =$$

$$= \frac{0.13 + 0.1 \times 0.5}{1 - 0.1 \times 0.5} = 0.189.$$

(59A)

Expression (59) also allows for deriving an expression for the risk premium:

- for the additive relation between the risk-free rate and the risk premium:

$$pr_{a2} = r_2 - r_f =$$

$$= \frac{r_f + p_d \cdot k}{1 - p_d \cdot k} - r_f =$$

$$= \frac{p_d \cdot k \cdot (1 + r_f)}{1 - p_d \cdot k} =$$

$$= \frac{0.1 \times 0.5 \times (1 + 0.05)}{1 - 0.1 \times 0.05} = 0.059,$$

(60)

- for the multiplicative relation between the risk-free rate and the risk premium:

$$pr_{m2} = \frac{1 + r_2}{1 + r_f} - 1 = \frac{p_d \cdot k}{1 - p_d \cdot k} =$$

$$= \frac{0.1 \times 0.05}{1 - 0.1 \times 0.05} = 0.053.$$

(61)

Consider the following calculations with the use of the IRA-2. The parameter values used will correspond to the previously adopted, or (for the new parameters will be used by other values. The order of values of the parameters will strictly correspond to the sequence of model parameters:

(49П)

(50П)